GEOMETRY

Exploring Centers

HENRY KRANENDONK AND JEFFREY WITMER

DATA-DRIVEN MATHEMATICS

DALE SEYMOUR PUBLICATIONS®

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About *Data-Driven Mathematics*

Historically, the purposes of secondary-school mathematics have been to provide students with opportunities to acquire the mathematical knowledge needed for daily life and effective citizenship, to prepare students for the workforce, and to prepare students for postsecondary education. In order to accomplish these purposes today, students must be able to analyze, interpret, and communicate information from data.

Data-Driven Mathematics is a series of modules meant to complement a mathematics curriculum in the process of reform. The modules offer materials that integrate data analysis with high-school mathematics courses. Using these materials will help teachers motivate, develop, and reinforce concepts taught in current texts. The materials incorporate major concepts from data analysis to provide realistic situations for the development of mathematical knowledge and realistic opportunities for practice. The extensive use of real data provides opportunities for students to engage in meaningful mathematics. The use of real-world examples increases student motivation and provides opportunities to apply the mathematics taught in secondary school.

The project, funded by the National Science Foundation, included writing and field testing the modules, and holding conferences for teachers to introduce them to the materials and to seek their input on the form and direction of the modules. The modules are the result of a collaboration between statisticians and teachers who have agreed on statistical concepts most important for students to know and the relationship of these concepts to the secondary mathematics curriculum.

A diagram of the modules and possible relationships to the curriculum is on the back cover of the Teacher's Edition.

Using This Module

"The earth is the center of the universe!" To 16th century thinkers, this statement seemed perfectly logical and correct. Today we chuckle at the absurdity of their thinking. We now know the idea of the earth as the center of the universe is a far more complex issue than previously thought by earlier scientists. Yet, centers as a concept of a special location remains an important idea. Shopping areas are planned and located on the basis of a center; airports are designed as "hubs" (or centers) for which other connections originate; schools, theaters, hospitals, and cultural attractions are selected and located as the "centers" for related activities.

Center, however, is not only a location. It also represents a type of thinking or a type of "balancing" of views and ideas. A politician might be called a "moderate" as she or he is often characterized as taking a central position on important issues. A judicial decision in a court of law might be based on balancing the issues of diverse groups. A principal in a school might balance the requests of various school groups by finding a compromise or center of agreement.

The physical concept of center is also an important consideration. Just ask a tightrope walker or a gymnast or a volleyball player about a center of balance. Without thinking about this center, the walker would fall, the gymnast would tumble, and a volleyball would dart in the wrong direction!

As you begin this study of centers, think of all the ways this word and its implications are used. Begin to collect newspaper articles, pictures, and objects to demonstrate ideas related to center. If your understanding of a center is unclear, then read further. This module will expand on the meaning of center and will hopefully give you new ideas about a familiar word. Stay tuned—your center is about to change!

Mathematical Content

This module covers such geometric and mathematical topics as medians of triangles, convex versus concave polygons, balancing points, fulcrums, centroids, center of mass, and coordinate systems on a plane.

Statistical Content

You will learn about the various kinds of centers, including mean, median, mode, and midrange, as well as weighted averages and "minimax" strategies that guard against the "worst-case" examples.

Unit I

Estimating Centers of Measurements

When is it okay to estimate the length and width of a room or to estimate the measure of a person's height?

When is it important for the measurements of a room to be "accurate" or the measurement of a person's height to be "accurate"? Is 1.65 meters a more accurate measurement of a person's height than 1.59 meters?

What is an "accurate" measurement?

Accuracy, measurements, and estimating are activities involved in this first section of *Exploring Centers.* As you read and work with each of the problems of this section, discuss what you would do to determine an accurate measure of your height.

Determine your height. Is it accurate?

LESSON 1

Centers of a Data Set

What is meant by a "center"?

How is a "center" calculated and interpreted?

The number representing your grade point average is a center, as well as the average number of students in a class or the average number of students arriving late to school. Would you consider the location of your locker as "centrally" located?

How would you determine this central location provided that was an important concern?

Although several examples of centers will be defined as these lessons are developed, examples based on the *best* estimate of a set of data will start the discussion of centers. The importance of this type of center is demonstrated using the backup times collected at a swim meet. Whenever possible, an electronic clock is used to record a swimmer's official time. If, however, this clock does not function correctly, which can easily happen due to equipment problems or a poor "touch" by the swimmer, three backup timers are used. Surprisingly, it is rare to have the same times reported by the three backup timers. If three backup times are used, how do you think the official time is determined for a swimmer?

OBJECTIVES

Calculate the mean, median, and mode of a data set.

Interpret a center as an estimation of a data set.

Visualize the accumulation of error resulting from estimates of measured data.

INVESTIGATE

Swim Meets

At a regional swim meet, Kristin, Melissa, and Shauna were involved in the same heat of the 50-yard freestyle. The top swimmers from each heat moved to the next level of competition. Their official times were used to both qualify them for the next competition and provide their placement in a heat of this event. As a result, the official time for each swimmer was very important! Although an electronic clock was used to record the official times at this particular meet, the following times were recorded by the backup timers:

Event: Girls' 50-yard Freestyle

Swim Club Record: 27.0 seconds

(All times recorded from the backup timers are in seconds.)

	Kristin	Melissa	Shauna
Timer 1	27.2	27.4	27.3
Timer 2	27.3	27.4	27.4
Timer 3	27.1	27.0	27.1

Discussion and Practice

1. Of the three swimmers, who probably won this swim heat? Why?

2. Of the three swimmers, who probably lost this heat? Why?

3. If the criterion for determining an official time was the best time recorded by the three timekeepers, who would benefit the most?

4. Do you think selecting the fastest time from the three times recorded for each swimmer is a fair method for determining an official time? Explain why or why not.

5. What method would you suggest for determining an official time if backup times were needed? Explain why you think your suggestion is fair.

6. Do you think the times reported for Melissa are unusual? Explain.

Mean, Median, and Mode

An official time for each swimmer could be determined in several ways. Three frequently used summaries for a data set are referred to as ***mean***, ***median***, and ***mode***. Each of these three summaries should be considered when selecting a center or description of a data set.

The ***mean*** of a data set is also called an ***arithmetic average***. It is calculated by finding the sum of the data and then dividing by the number of members in the set. It is the most common summary of a set of numbers.

7. The following process determines the mean of the three times reported for Melissa:

$$\frac{(27.4 + 27.4 + 27.0)}{3} = \frac{81.8}{3} = 27.2\overline{6}$$

a. Describe how a mean represents a "center" value of this data set.

b. Does the mean seem to be a fair value of Melissa's swim time?

c. Would you recommend the mean as the official time of a swimmer if the electronic time was not accurate? Explain.

The ***median*** is another summary of data. Generally it is described as the "middle value" of an ordered data set. This middle value is most easily determined if the number of values belonging to the data set is odd. In those cases, the median is an actual value of the data set. If, however, the data set has an even number of values, then the median is the mean, or arithmetic average, of the two data values "centered" around the middle of the set. Following is an example of finding the median for a data set of six values:

Data set: 34, 42, 16, 30, 40, 45
Ordered data set: 16, 30, 34, 40, 42, 45

The median ("middle" value) is the mean of the third and fourth values:

3rd value 4th value

16, 30, 34, 40, 42, 45

$$\frac{(34 + 40)}{2}$$

$$= \frac{74}{2} = 37$$

8. Similar steps are used to find the median of Kristin's three times. Arrange (27.2, 27.3, 27.1) in ascending order:

27.1 27.2 27.3

The middle or median is the second member of the ordered values:

27.1 27.2 27.3

↑
The median

a. Describe how a median represents a "center" value of this data set.

b. Does the median seem to be a fair representation of Kristin's time?

c. Examine Melissa's and Shauna's recorded times also. Would you recommend the median as the official time of a swimmer? Explain.

9. Calculate the mean of Kristin's times.

a. What is unusual about Kristin's data set when finding the mean and median?

b. Why does this happen in part a?

The ***mode*** is defined as the "most frequent data value" or values. A mode exists only if a specific data value in the data set occurs more than once. If several values reoccur in an expanded set of numbers, then the mode is the most frequent value. It is possible for a data set (with more than three values) to have more than one mode. When all data values occur only once in a data set, then the data set is considered to have no mode.

10. Of the three swimmers, who would be able to have her official times summarized by a mode?

11. The mode does not actually describe a "center" value for a data set of three swim times. Explain this statement by using the times for one of the swimmers.

12. If a mode exists for a data set of three values, it is similar to what other data summary, the median or mean? Demonstrate this by developing another set of three numbers.

13. Consider a set of four or more numbers in which at least one mode exists. Would the mode of this expanded set of numbers always have the same feature you summarized above? Explain your answer or demonstrate with a data set.

14. Generally swim meets use the median time as the official time of a swimmer when a backup time is needed.

   a. Why do you think this value is used?

   b. Officials attempt to obtain three backup timers. Why? How would the situation change with two timers, assuming they use the median? How would it change with four timers?

Centers of Measurements

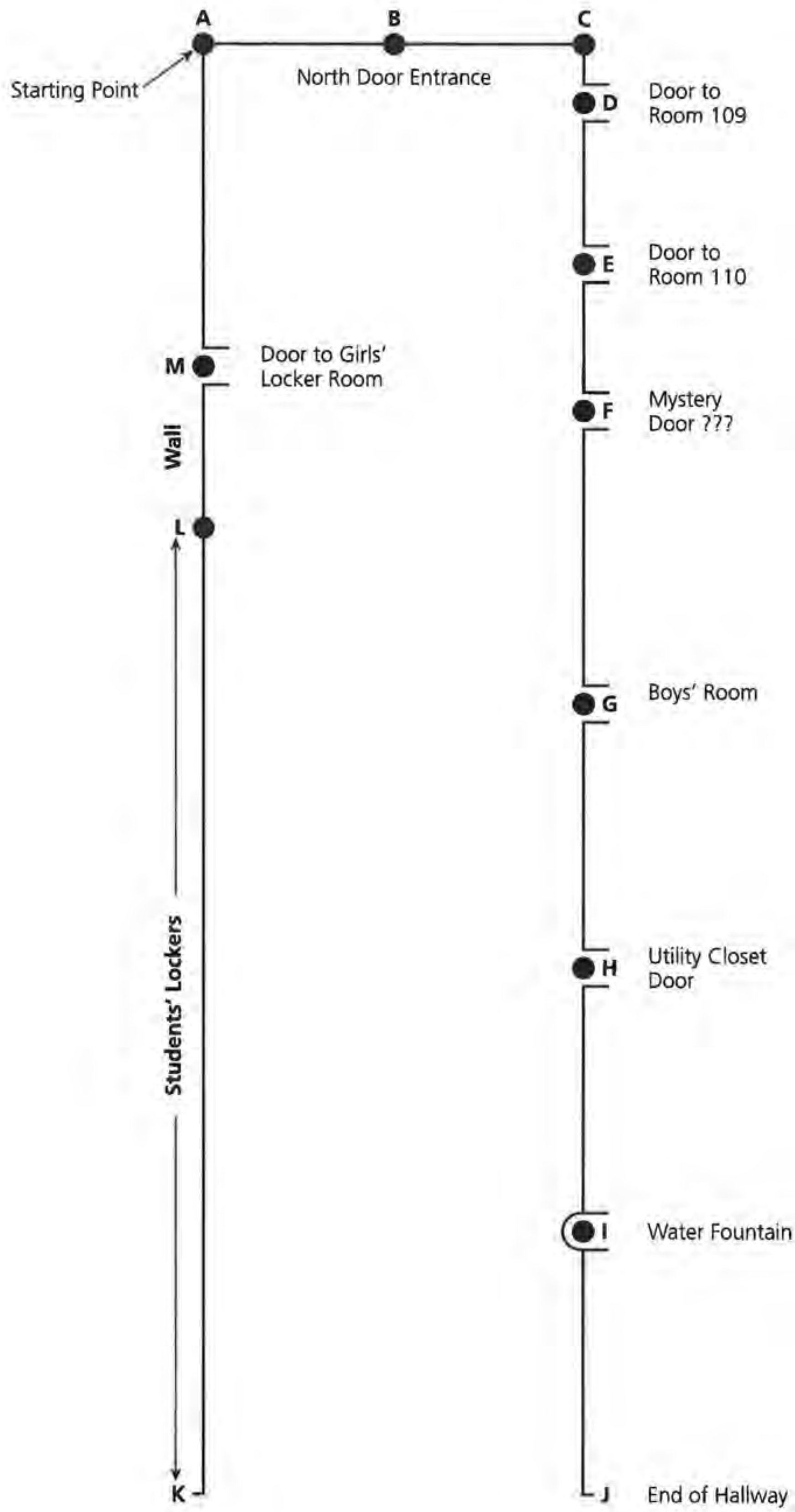
Students in a geometry class at Rufus King High School were asked to develop a scale drawing of the first floor north entrance hallway. This drawing was to be used to determine measurements and calculations for purchasing floor tiles, bulletin boards, and so forth. Estimates of costs for each of these projects require an accurate drawing of this hallway. This project was developed by the class in the following way:

Step 1. Key locations along the hallway floor were marked and labeled with masking tape.

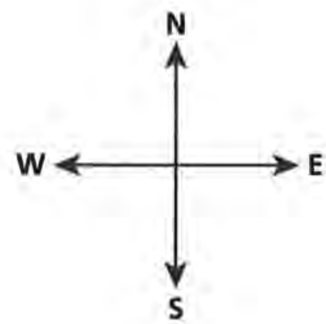
Step 2. A rough sketch of the hallway was developed to highlight the key locations.

Step 3. Seven groups of students were formed. Each group was responsible for recording the measurements obtained on the Data Summary Sheet.

Step 4. Using the data collected from the seven groups, each group was responsible for developing a scale drawing of this rectangular hallway.



This sketch was developed by a teacher to provide relative positions of the key locations involved in this project. Remember, this drawing is a rough sketch and should not be used in making estimates of the actual distances! The hallway is a rectangle.



Data Summary for the Measurement Experiment
Rufus King High School North Entrance Hallway (meters)

Group	AB	BC	CD	DE	EF	FG	GH
Group 1	1.78	1.77	0.65	8.34	1.87	5.02	2.10
Group 2	1.78	1.75	0.58	9.35	1.85	4.98	1.75
Group 3	1.77	1.75	0.59	9.20	1.88	4.95	2.25
Group 4	1.78	1.76	0.66	9.10	1.90	5.03	1.95
Group 5	1.75	1.80	0.67	9.05	1.89	4.95	2.23
Group 6	1.68	1.76	0.63	9.10	1.88	4.80	2.15
Group 7	1.78	1.76	0.66	8.91	1.87	5.10	2.17
Mean							
Median							
Mode							

Group	HI	IJ	JK	KL	LM	MA
Group 1	3.34	3.08	3.55	8.88	5.30	11.32
Group 2	3.96	3.50	3.65	9.52	5.00	10.91
Group 3	3.55	3.64	3.56	9.14	5.62	12.12
Group 4	3.12	2.68	3.72	8.78	5.45	11.52
Group 5	3.05	2.85	3.43	8.68	5.15	13.58
Group 6	3.22	2.98	3.31	8.94	5.23	9.12
Group 7	3.40	3.05	3.33	11.25	5.35	11.56
Mean						
Median						
Mode						

15. Using the class measurements, answer the following questions before developing a scale drawing.

a. Review the data sets. Each group was expected to measure the same distances. Why are the recorded measurements different?

b. Complete the data sheet by determining the means, medians, and modes for each of the distances labeled. Copy and complete this part of the data sheet.

c. You will be developing a scale drawing of this rectangular hallway. What do you anticipate to be the main problem in constructing an accurate sketch of this hallway?

16. Consider the following four criteria in selecting the "best" center of the measurements reported by the groups.

- the mean of the values for any of the specified distances
- the median of the values

- the mode of the values (if one exists)
- an "average" (mean or median) of a subset of the values

(The last criterion allows some measurements to be thrown out as obvious errors. The measurements remaining would then be averaged or "centered.")

a. Determine the measurements you will use to develop the scaled sketch of the rectangular hallway. Explain what criterion you used to select this best estimate and why. Complete the following table which is also on *Activity Sheet 1*.

Measurement of segment to be used in your sketch	**Criteria used for this measurement**	**Why?**
AB =	_______	_______
BC =	_______	_______
CD =	_______	_______
DE =	_______	_______
EF =	_______	_______
FG =	_______	_______
GH =	_______	_______
HI =	_______	_______
IJ =	_______	_______
JK =	_______	_______
KL =	_______	_______
LM =	_______	_______
MA =	_______	_______

b. Develop a scale drawing of this hallway based on the measurements selected in your table. Begin developing this scale drawing by placing the starting point **A** in the upper-left corner of a blank sheet of paper. (You might consider developing this sketch on legal size paper as this hallway is quite long.) Measure and mark each of the labels provided in the teacher's sketch of this hallway. Use a scale of **1 cm = 1 meter** or a comparable scale.

17. Your drawing represented the best measures of the distances given your criteria. As you compare your sketch to other members of the class, how does each sketch indicate the measures are not "perfect"?

18. How could the fact that the hallway is rectangular help in developing an accurate sketch?

19. Of the measurements recorded on the data sheet, what specific distances do you think are the most likely to be inaccurate. Explain why you think this.

20. A group of students studying the collected data commented that the Rufus King students obviously used meter sticks (rulers) instead of a measuring tape. Would you agree? Explain your answer.

SUMMARY

Collected data frequently needs to be summarized by a center. Mean, median, or mode are summaries of a data set representing its "center" or summary.

Practice and Applications

21. From your sketch, determine the following distances:

a. the distance (to the nearest meter) from the door of room 109 to the girls' locker room door.

b. the distance from the water fountain to point **A**.

c. the distance from the boys' room door to the girls' locker room door.

22. If floor tiling will be installed at a cost of $16.75 per square meter, what is your estimate of the total cost of installing floor tiles for the entire hallway?

23. An announcement board for the athletic department will be installed from point **A** to a point 1 meter north of the girls' locker room door (or 1 meter north of point **M**). If your scale drawing is used to estimate the length of the board, what problem do you encounter with this last segment of the drawing?

24. If the announcement board discussed in problem 23 costs $10.70 per linear meter, what is the projected total cost of this project? (The announcement board has a standard height, therefore, your estimates do not need to involve that dimension of the board.)

The following suggested problems follow the process described in the investigations of this lesson by the Rufus King students.

25. As a class, identify a hallway or room in your school that could be used to develop a scale drawing.

a. Select key locations along the perimeter of this room or hallway. Either with masking tape or paper, identify the key locations.

b. Develop a sketch of the room or hallway using the key locations.

c. Design a data sheet to record the measurements indicated in the sketch.

26. Form several groups to determine measures of the distances. Each group should complete the following steps:

a. Using the same type of measuring tool (for example, meter stick or tape measure), measure and record the values designated in the class data sheet.

b. Estimate the distances designated in the sketch using some criterion of centering.

c. Design a scale drawing of the room or hallway.

27. Is the room or hallway selected for this class project a rectangle, square, or other shape?

How can the shape of a room help you determine the accuracy of the scale drawing?

28. How could each group test the accuracy of its scale drawing?

LESSON 2

Descriptions Through Centers

Have you ever watched athletes competing in track and field events?

What kind of special preparation would be necessary for competing in an event such as the broad jump?

Track and field events require athletes to coordinate running and jumping skills. Athletes participating in the high or low hurdles spend considerable time counting the number of steps needed to reach a position to begin the jump over a hurdle. Incorrectly counting the number of steps can throw off the coordination and the resulting time for the athlete to complete the event. Similarly, the broad jump event in track and field also requires athletes to coordinate running and jumping.

OBJECTIVES

Calculate the mean, median, and mode of a data set.

Interpret a center as an estimation of a data set.

Interpret a center as a description of an individual's physical characteristics.

INVESTIGATE

Track and Field Events

Olympian Carl Lewis carefully prepared for several summer Olympics (and subsequent world records in the broad jump) by running a specific distance and counting his steps or strides before making the actual jump. How might Carl determine the number of steps to the jumping-off line?

Discussion and Practice

1. Consider the broad jump event in a track and field meet.
   a. Why is it important to coordinate running and jumping with this event?
   b. What are some factors that might affect the number of steps or strides of a particular broad jumper?

2. How could you determine how many steps it takes you to walk a distance of 50 meters?

3. Jason wanted to know how many steps it would take him to walk a distance of 50 meters. He decided to estimate the length of one step by marking on the floor the starting and ending positions of his feet for one "typical" step. He measured this distance with a meter stick. Using this method, Jason described the length of his step as 85 centimeters. Use Jason's measure of 85 centimeters to answer the following:

   a. Determine an estimate for the number of steps Jason would take to walk a distance of 50 meters.

   b. Do you think your estimate in part a is accurate? Why or why not?

   c. How could you evaluate the accuracy of this estimate?

4. Select a few volunteers from your class and using Jason's method, measure the length of one step of each volunteer.

   a. Do any of your volunteers have a step measure greater than 85 centimeters? If yes, are there any additional descriptions or characteristics shared by these volunteers? Explain.

   b. Do any of your volunteers have a step measure less than 85 centimeters? If yes, are there any additional descriptions or characteristics shared by these volunteers? Explain.

If Jason were to measure his step again, it is very likely that he would record a value different than 85 centimeters. Why? Because there are variations involved in a "typical step," an average value might be the best way to describe the number of steps needed to walk this distance.

Estimate Steps Required for Specific Distances

This investigation involves estimating the number of steps members of your class would take to walk a distance of 50 meters or any other designated distance. Consider the following procedure:

- Select two or three students from your class to carefully measure a distance of 50 meters (or designated distance) in a hallway of your school by using a measuring tape or meter stick.

- Mark a convenient starting and ending point on the floor of the hallway with masking tape.
- When given permission by your teacher, as many members of your class as possible should **walk** (do **NOT** run, jump, or skip for now) the measured distance of 50 meters (or designated distance) and count the number of steps it took each person to reach the end position.
- If you walked this distance, record your count as the result for Trial 1. Repeat this procedure by walking this same distance for four additional trials. For each trial, record the approximate number of steps counted. (Do not worry about any fractional step needed to reach the end position. Use your best judgment to estimate whether or not to count the last step. Record all trials as whole numbers.)

Your Name	Trial 1	Trial 2	Trial 3	Trial 4	Trial 5
______	______	______	______	______	______

5. Did you expect to record the same number of steps for each of your trials? Why or why not?

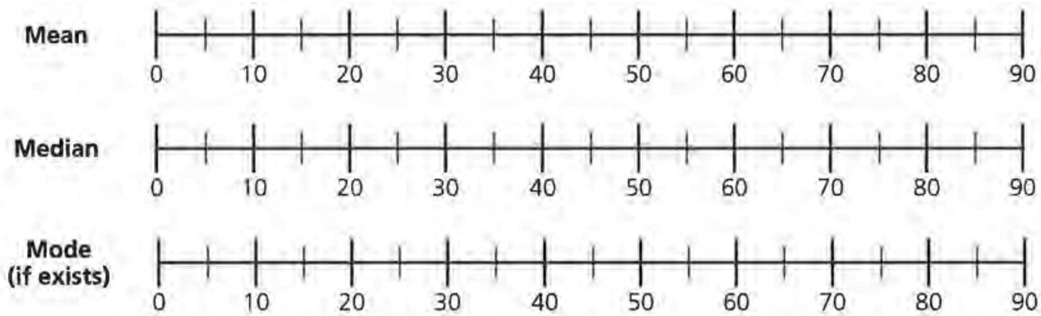
6. Conduct a poll of the students in your class who walked the distance described.

- **a.** How many students recorded the same number of steps for all five trials?
- **b.** How many students recorded a different number for each of the five trials?
- **c.** How many students had four trials the same?
- **d.** How many students had two trials the same?

7. Determine the following summaries of the five trials you recorded.

- **a.** the **mean** of the five trials
- **b.** the **median** of the trials
- **c.** the **mode** of the five trials (provided a mode value exists)

8. Develop a visual comparison of your mean, median, and mode by recording each average on a number line similar to the following:



9. This experiment was based on counting the number of steps needed to walk approximately 50 meters. How could you use these results to determine the length of one of your steps?

10. Based on the **mean** value of the five trials, determine the length of one of your steps.

   a. in meters.

   b. in centimeters.

11. Based on the **median** value of the trials, determine the length of one of your steps.

   a. in meters.

   b. in centimeters.

12. Based on the **mode** value of the trials (provided your data set had a mode), determine the length of one of your steps.

   a. in meters.

   b. in centimeters.

13. Which average do you think is the **best** description of your "typical step"? Why?

14. What might be a reason for using each of the following in this particular example?

   a. a mean as the best estimate

   b. a median as the best estimate

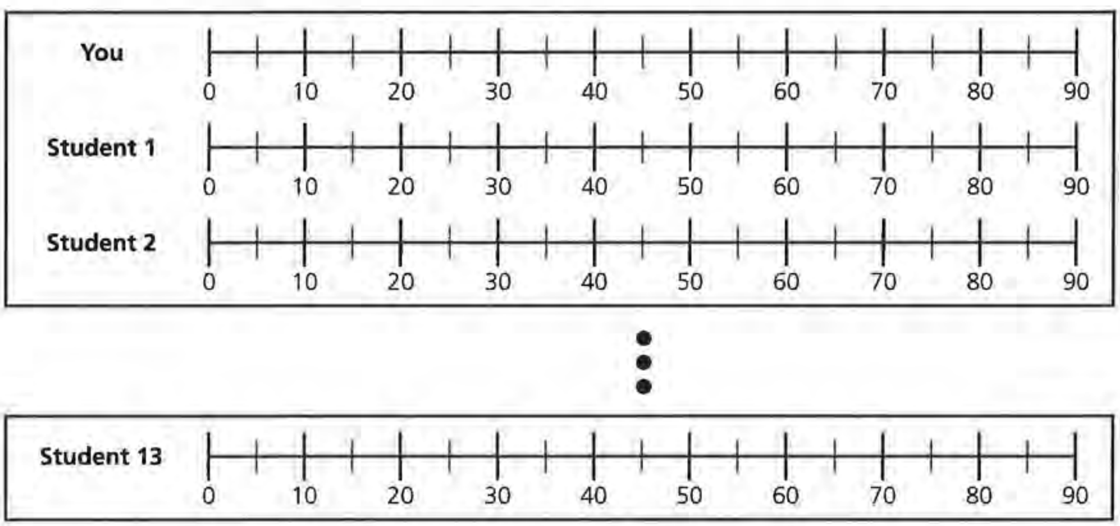
   c. a mode as the best estimate

15. Select mean, median, or mode. Collect at least 13 averages of this type from students in your class. In other words, collect 13 means or 13 medians or 13 modes from the other students involved in this project. Do not mix the type of averages collected from the other members of your class! Copy and complete the following table:

Average selected in your sample: ____________________
(Mean, median, or mode)

Name	Number of Steps
⋮	⋮

16. Visually represent the results of the 13 summaries you collected by plotting the value for each student on the number lines provided on *Activity Sheet* 2. Include your summary in this collection.



17. Examine the data collected from your class.

a. Identify the two students in your data set who recorded the fewest number of steps.

b. Identify the two students in your data set who recorded the greatest number of steps.

c. As you consider the students identified in parts **a** and **b**, are there any descriptions or characteristics that distinguish the students who recorded the fewest number of steps from the students who recorded the greatest number of steps?

18. Consider the following additional descriptions or characteristics of the students in your class.

- Month of their birthdays (January = 1, February = 2, etc.)
- Height (in inches or centimeters)
- Length of their forearms (in inches or centimeters)
- Shoe size
- Number of brothers and sisters
- Circumference of wrists (in inches or centimeters)
- Number of sit-ups completed in 30 seconds

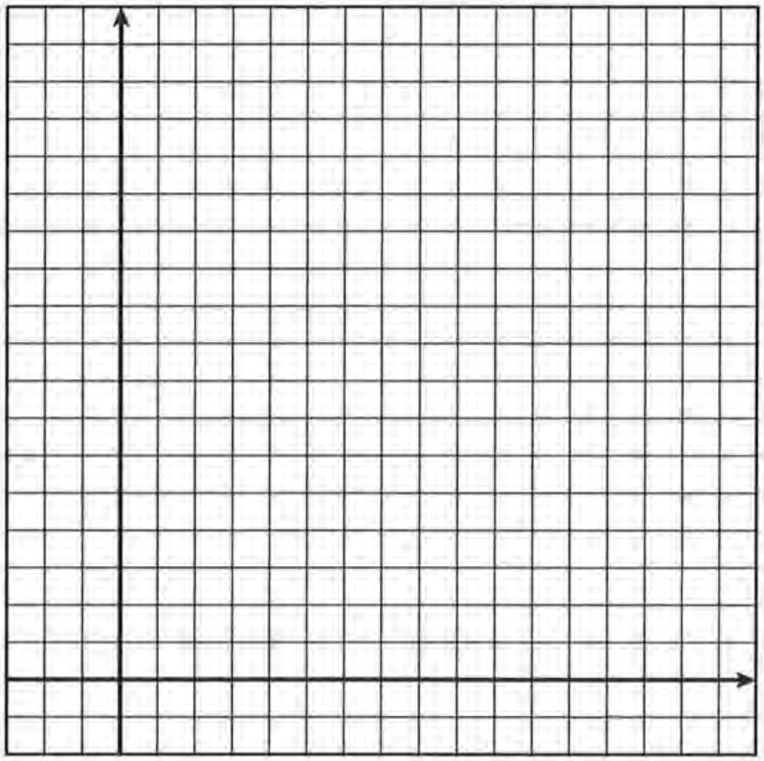
a. Which descriptions or characteristics in the list above do you think would **not** distinguish the students who recorded the fewest number of steps from the students who recorded the greatest number of steps?

b. Select one of the additional descriptions or characteristics you think might distinguish the two groups of students and explain why you selected this item.

c. Collect from the 13 students in your sample the value corresponding to the characteristic you selected in part **b.** Organize this additional piece of data in a chart similar to the one below.

Student	Additional Description (x-value)	Number of Steps (y-value)

19. Using the x- and y-values as indicated in the chart, plot the points representing your sample of students on a coordinate grid similar to the one below. You may use *Activity Sheet 3*.



20. Do the points you plotted in problem 19 indicate a relationship between the Additional Description and the number of steps? If so, describe the relationship.

21. If you were allowed to run the 50-meter distance, how would the number of strides counted in this distance change?

SUMMARY

A value used to describe the length of a person's step or the number of steps needed for a person to walk a specific distance may best be described as a center of a data set. A mean, median, or mode can be used to identify the center.

When two variables are investigated, a coordinate grid may be used to visualize their relationship.

Practice and Applications

22. Corey Reed collected the following data or number of steps taken as he walked a distance of 30 yards (note: this distance is measured in yards):

42, 41, 44, 43, 42, 45

a. Determine the mean of Corey's data set:

b. Determine the median of Corey's data set:

c. Determine the mode of Corey's data set:

23. Corey also collected the height and mean number of steps for several other students in his class. He recorded his collected data in the following chart:

Name	Number of steps (x)	Height in inches (y)
Sam Bland	37	74
Robert Rotella	38	76
Claud Salyards	42	68
James Rockweger	41	41
Devon Brooks	40	72
Anthony Balistreri	45	67
Rachel Jones	44	64
Jeffrey Durr	43	70
Laura Frye	42	69

Study the data set recorded by Corey. He might have erred in measuring! Corey most likely erred in measuring the height of which person? Explain why you think this measurement (or measurements) was incorrect.

24. Develop a **scatter plot** of the values indicated in the chart of problem 23 on a coordinate grid provided on *Activity Sheet* 3. DO NOT include any values from the data set you think represent an error in Corey's measurement.

25. Does height seem to be related to the mean number of steps recorded to walk 30 yards? Why or why not?

26. Estimate James Rockweger's height. How did you determine this estimate?

27. Estimate Corey's height. How did you determine this estimate?

Centers of Balance

Does your school determine a grade point average for each student? Many schools base a grade point average on a 4-point scale. If that scale is used, what does a 3.5 mean?

A certain student received 3 As, 2 Bs, and 1 C. How would you determine this student's grade point average?

A bowler averaged 130 per game (or round) for the last 8 games in a tournament. If there are 2 games left to play, is it possible for this bowler to complete the tournament with an average score of 135? If yes, what combined score must this bowler get for the last 2 games? Do you think this is likely to happen?

Averages describe centers of a data set. What is special about this center? How is it used to understand the data? The material in this section of *Exploring Centers* will specifically investigate the mean and its application.

As you study this section, you are to collect data from each member of your class or from at least 50 members of your school that can then be used to determine the following:

- **What is the average age of the students in your class or sample?**
- **What is the average height of the students in your class or sample?**
- **What is the average number of days students in your class or sample were absent during a specific two-week period?**
- **What is the average number of hours students in your class or sample worked on school material at home during a specific two-week period?**
- **What is the average number of brothers and sisters for students in your class or sample?**

Before you begin, think about this: In what ways are the above averages useful?

LESSON 3

The Mean as a Center of Balance

In what situations is balance important?

Have you ever tried walking on a balance beam?

What could you do to improve your balance on the balance beam?

A keen sense of balance is often important in athletics. After watching the incredible movements of a gymnast on a balance beam, you may begin to appreciate the skills of balance necessary just to stay on the balance beam. This athlete's maneuvers, however, can also be interpreted from a science perspective. Studies of a gymnast's movements and sense of balance involve an intricate study of centers of balance.

OBJECTIVES

Determine the mean of a set of points presented on a number line.

Interpret the mean as a center of balance.

INVESTIGATE

Balance

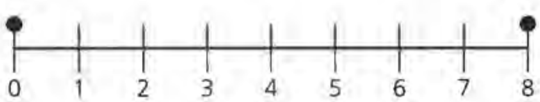
A discussion of balance begins with a basic concept of distributing weights. To illustrate this concept, consider the following model: two raisins of equal weight are taped to the ends of a lightweight ruler. The goal is to balance the ruler with the attached raisins on the broad side of a pencil. (Note: if you attempt to develop this model, a lightweight, flat ruler will approximate the results developed in this section.)

1. What variables affect the ability of the ruler to balance the raisins?
2. Move the broad side of the pencil underneath the ruler until a position is found that balances the ruler with the attached raisins. Where did you locate the pencil relative to the attached raisins to achieve a balance? Was this hard to do?
3. Would replacing the two raisins with two heavier objects (each weighing approximately the same) affect the balance? Try it by replacing the raisins with golf balls or similar objects.
4. Retape raisins to two other positions along the ruler. How would you change your estimate of where the pencil should be located to balance the ruler? Try and see if your estimate balances the ruler with the raisins.

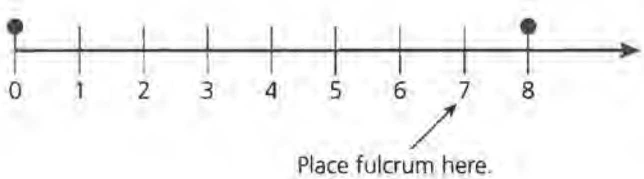
The pencil represents a support used to balance the ruler. This support is called the ***fulcrum*** of the model. The challenge is to position the fulcrum (or support) so that the ruler will balance on the pencil.

Center of Balance on a Number Line

Consider the ruler to be weightless. If that were possible, the weight of the raisins and the location of the raisins along the ruler determine the position of the fulcrum. The following number line can be used to represent the ruler and raisins:

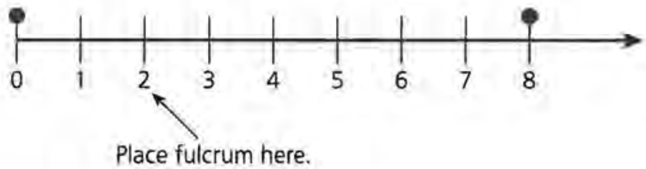


5. Consider taping raisins along the number line as illustrated. Place a fulcrum or support at various locations along the number line. Sketch the number line if:
   a. the fulcrum or support were placed at location 7 on this number line.



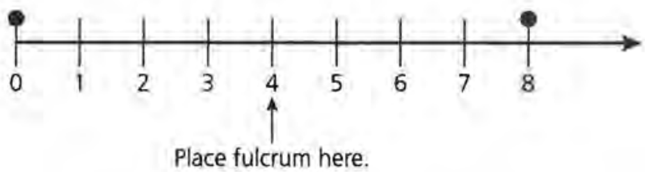
Sketch ruler with fulcrum at position "7".

b. the fulcrum or support were placed at location 2 on this number line.



Sketch ruler with fulcrum at position "2".

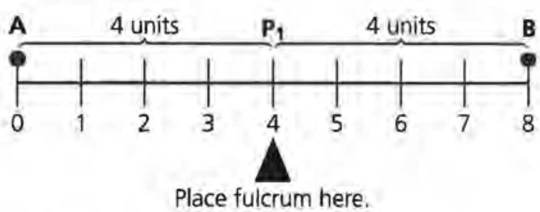
c. the fulcrum or support were placed at location 4.



Sketch ruler with fulcrum at position "4".

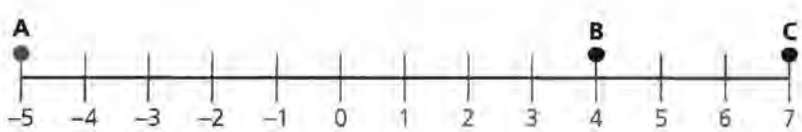
This last example balances the placement of the attached raisins. Similar to the situation observed when the pencil balanced the ruler with raisins located at two locations, placing a fulcrum in-between the weights as illustrated balances the arrangement of the weights.

The distance from 0 to 4 is 4 units; similarly, the distance from 8 to 4 is 4 units. Therefore, the balance point is the location in which the distances to the right and left of the balance point are equal. Represent this balance point as $\mathbf{P_1}$.



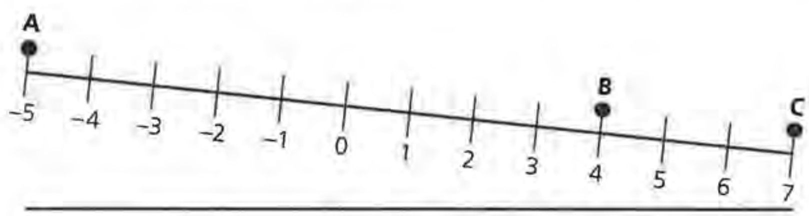
6. What descriptions of **center** could be used to summarize P_1?

Recall, each point identified on the number line represents a raisin or "object of equal weight." Each object is also called a "**point mass**" when describing this type of problem. Suppose there are three raisins on an expanded number line located at **A, B,** and **C** as shown below.



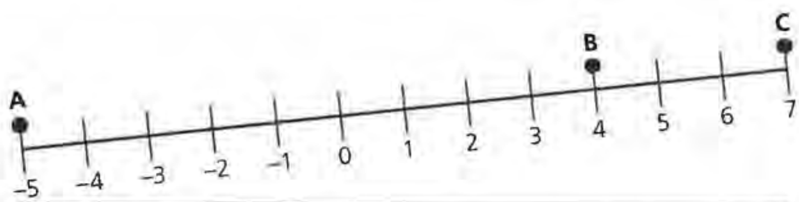
(Again, assume the number line is weightless. Obviously the ruler you used to demonstrate this model had weight and contributed to the location of the balance point. As the weight of the ruler is uniformly distributed, the weight and position of the raisins along the ruler essentially determine the balance point of each model.)

7. Where could the fulcrum be placed to produce the following sketch of the number line?



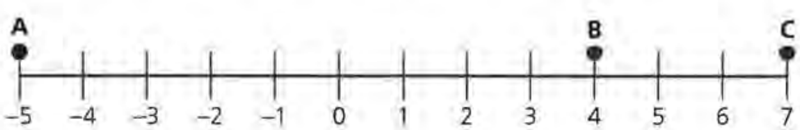
There could be several locations of the fulcrum that could produce the effect as illustrated by the above sketch. Copy the above sketch and shade the possible locations of the fulcrum that would result in the lack of balance as illustrated. Try modeling this sketch with a ruler and three raisins.

8. Where could the fulcrum be placed to produce the imbalance as illustrated in the following diagram?



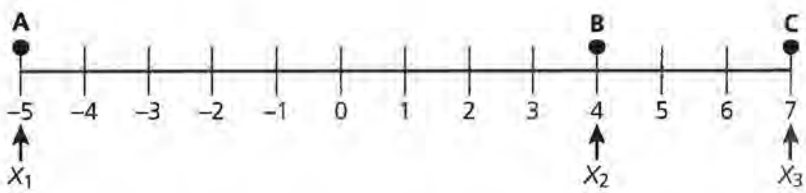
Again, there could be several locations of the fulcrum that could produce the effect illustrated above. Copy the above sketch and shade the locations of the fulcrum that would result in the above imbalance of the number line. Also, experiment with a ruler and raisins to test your estimates.

9. Before you calculate the location of the fulcrum that would balance the three raisins, estimate where you would place the fulcrum to balance the model shown below.



When two raisins were placed on the number line, the balance point was the location halfway between the weights. This location is also the ***mean*** or arithmetic average of the values representing the positions of the raisins. The mean centers the distances to the right and to the left of the location of the fulcrum. Can the mean similarly balance the distances for the example involving three points on the number line?

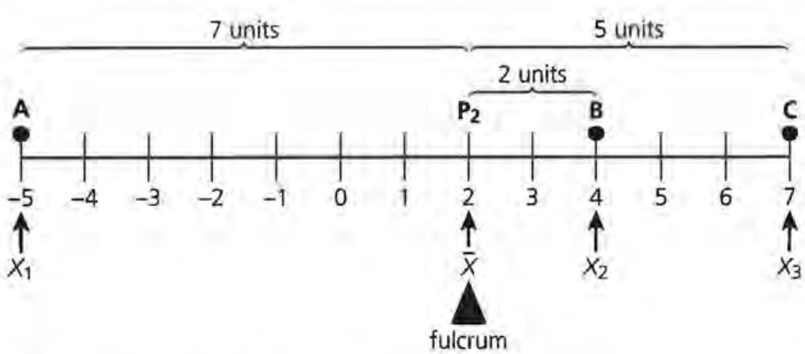
Let the value of each point on the number line be represented as x_1, x_2, and x_3. This arrangement is summarized by the following diagram.



Represent the position of the fulcrum for weights placed at points A, B, and C as P_2. What if the value of P_2 were estimated as $\bar{x}$, the mean of the values of the points? For this example, $\bar{x}$ would be:

$$\bar{x} = \frac{-5 + 4 + 7}{3} = 2$$

$$\bar{x} = 2$$



10. According to the diagram, how many units is

a. B from P_2?

b. C from P_2?

c. A from P_2?

11. Examine again the previous diagram involving three points on the number line.

a. What point or points representing raisins are located to the right of P_2?

b. Determine the total distance from P_2 to the point or points to the right of P_2.

c. What point or points are located to the left of P_2?

d. Determine the total distance from P_2 to the point or points to the left of P_2.

12. What characteristic of the mean balances the distribution of the raisins?

SUMMARY

Objects of ***equal weight*** positioned on a number line have a center of balance located at the ***mean*** of the locations of the objects. Placing a fulcrum at this center balances the total distances of the objects to the right of the fulcrum with the total distances to the left of the fulcrum. Balance is maintained by this equal distribution of the distances from the fulcrum.

Practice and Applications

13. Draw a number line as shown. Locate and label points A, B, and C. Let each point represent the position of an object of equal weight on the number line.

A = –5
B = –2
C = 7

14. Determine the location of the center of balance for the points specified in problem 13. Label the center of balance on the number line you designed as P.

15. For each of the following, construct a number line and plot the points indicated. Determine the center of balance for each example (label the center of balance on the number line as point P).

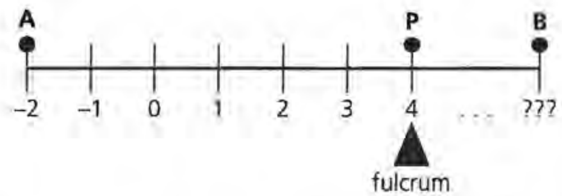
a. A = –3, B = 0, and C = 6

b. A = –8, B = –7, and C = 6

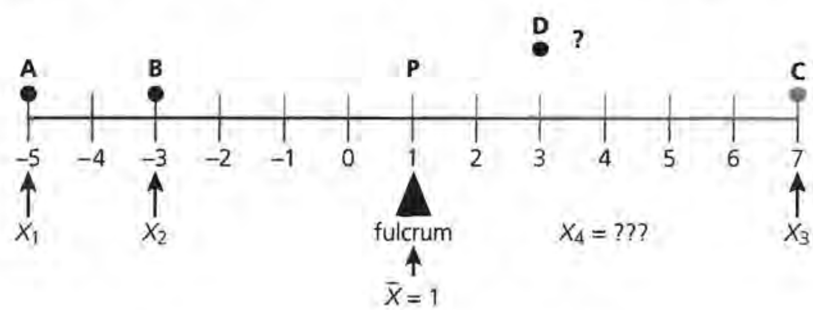
c. A = –4, B = 2, C = 5, and D = 8

d. A = –5, B = –3, C = 0, and D = 9

16. The location of the fulcrum P that balances two objects of equal weight placed at positions A and B is 4. You know A is located on the number line at position –2. Determine the location of point B. Explain how you determined your answer.



17. Four points are placed on a number line so that the center of balance is located at 1. The positions of 3 of the points A, B, and C are labeled on the following number line.



Use the following method to locate the fourth point, D.

a. Rewrite the following using a value or variable represented in the illustration to replace each "??".

$$\bar{x} = \frac{-5 + -3 + 7 + \boxed{??}}{4} = \boxed{??}$$

b. Solve the above for the location of point D or x_4 on the number line.

18. Consider the following two examples. Determine the missing values and plot the results on a number line for each.

Point A x_1 =	Point B x_2 =	Point C x_3 =	Point D x_4 =	Location of the Balance Point P $\bar{x}$ =
–5	–3	0	3	?
–4	2	?	4	1

19. Brian had a mean of 20.5 points on four 25-point quizzes in physics. He recalled his scores on three of the quizzes. They were 18, 24, and 22.

a. Without actually calculating Brian's score on the fourth quiz, determine if this score is greater or less than his average of 20.5. Why?

b. Determine the specific value of Brian's fourth quiz score.

c. Represent Brian's scores and mean on a number line similar to the examples presented in this lesson.

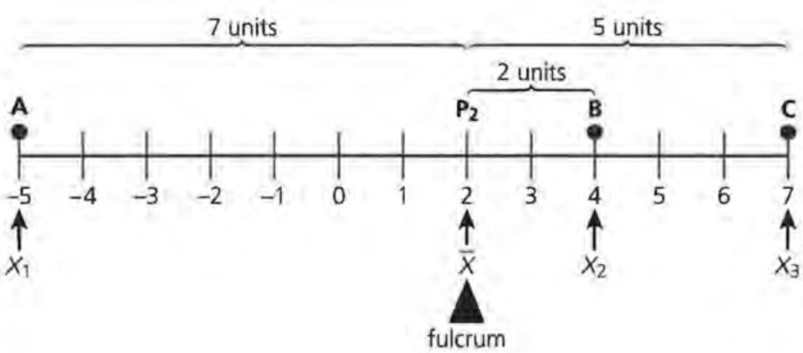
LESSON 4

Weighted Averages

What if the objects located along a number line are not of equal weight?

Suppose a weight at point A in the diagram below weighed more than the weights at points B and C. In what way would this change the location of a fulcrum representing the balance point?

How would you locate a balance point if the weights located along the number line were not of equal weight?



The number line illustrated in the previous lesson provided a visual way to represent the positions of objects of equal weights. The location of a balance point for these objects describes an important point, or center. If the number line is weightless and the objects located along the number line are of equal weight, then the "balance" point or center is where the total distances from the objects to the right of this center are "balanced" by the total distances from the objects to the left.

OBJECTIVES

Determine the center of balance by calculating the weighted mean of objects located along the number line.

Determine the weighted mean for problems of percents and similar applications.

INVESTIGATE

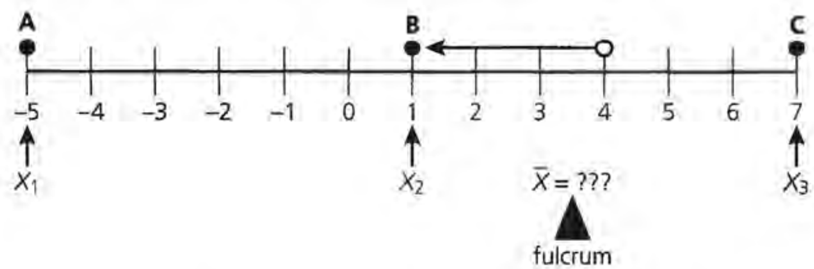
Balance Points

In the previous lesson, three raisins were located along the number line at the locations indicated as A, B, and C. Return to your ruler and raisins. Tape three raisins along the light-weight ruler. Use the broad side of a pencil and find a position of the pencil that balances the ruler and the raisins. Identify one raisin as **A**, the middle raisin as **B**, and the third raisin as **C**.

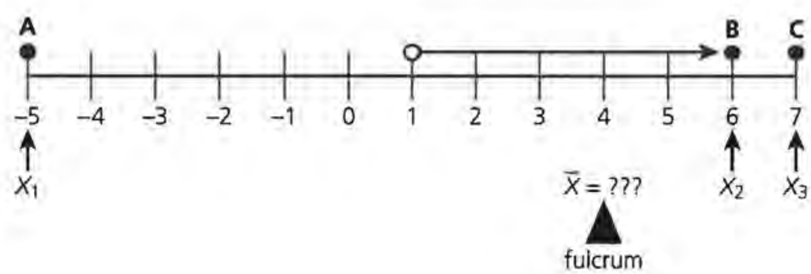
1. In which direction does the location of the fulcrum change as the raisin or weight identified at **B** is moved toward **C**? Try sliding a raisin represented in your model toward the raisin located at **C**. Now readjust the pencil representing the fulcrum. In what direction did you move the pencil to balance the new arrangement?

2. Consider each of the following changes to the locations of the raisins.

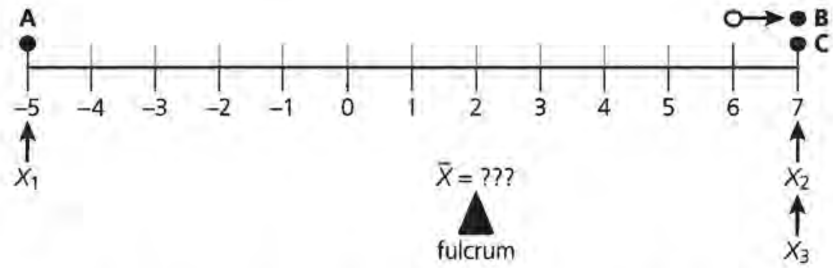
   a. Determine the new location of the fulcrum to balance the three raisins if the weight located at **B** is shifted to the position 1 on the number line.



   b. Determine the location of the fulcrum to balance the three raisins if the weight located at **B** is shifted to the position 6 on the number line.



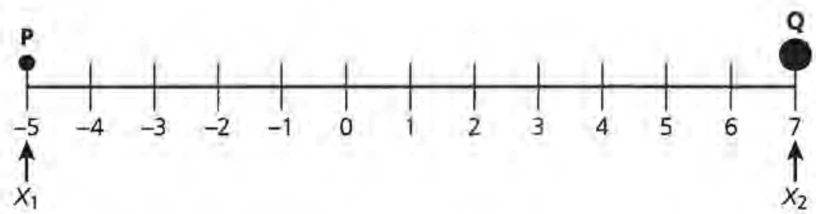
3. Continue shifting weight B to the right until it is located at position C as shown.



a. What is the mean of this arrangement of the three raisins?

b. Do you agree or disagree with this statement: "The balance point of the 3 raisins shifts in the same direction as the shift of weight B." Why or why not?

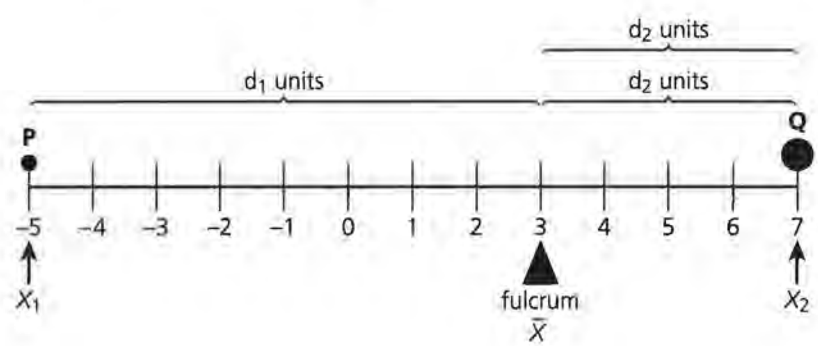
Consider a weightless number line with two weights **P** and **Q** attached at the locations indicated. Consider the weight at location **P** to be equal to the weight of one standard raisin. Consider the weight located at **Q** to be equal to the weight of two standard raisins.



4. Consider the following.

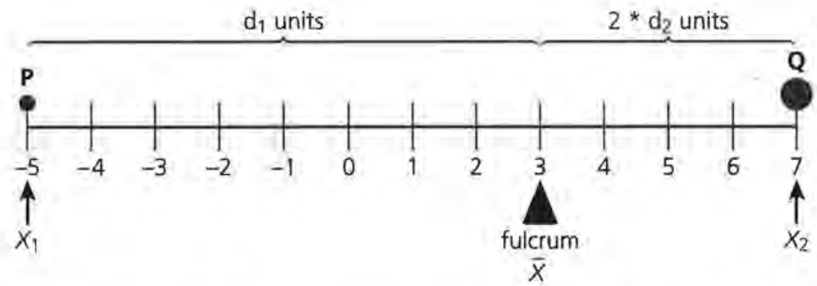
a. Is this the same problem as presented in problem 3? Why or why not?

b. Determine the values of d_1 and d_2 by considering two raisins at location **Q**.



c. Are the total distances to the right of the fulcrum balanced by the total distances to the left?

Another way of dealing with this picture would be to take into account the combined weights of the raisins at location Q. As the weight represented at that location is twice the weight represented at location P, the distance from the balance point to Q is multiplied by 2. This is illustrated in the following diagram:



5. Organize the data from this number line by completing the following table:

Point	Weight W_i	Position on the number line x_i	Weighted value $W_i x_i$
P	______	−5	______
Q	2	______	14

6. Use the table developed in problem 5 to answer the following:

a. Why is x_i (position) multiplied by W_i (the weight of the raisin) located at that point?

b. What is the sum of the values calculated under the column heading "Weighted value" or "$W_i x_i$"?

c. What is the sum of the values under the column heading "Weight" or "W_i"?

d. What is the sum of $W_i x_i$ divided by the sum of W_i?

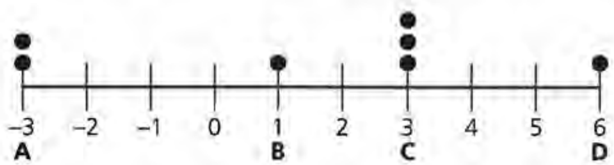
SUMMARY

The balance point of weights distributed along a number line is determined by the **weighted average.** The weight of each object is multiplied by its position on the number line to represent its weighted value. The sum of each value is then divided by the total weight of the objects to determine the weighted mean.

This is summarized by the following formula:

$$\bar{x} = \frac{\text{sum of } W_i x_i}{\text{sum of } W_i}$$

Consider the following weightless number line and objects of equal weight (for example, raisins) located on this number line.



7. Copy the above diagram.

a. Without actually calculating, label your estimate of the balance position as point E.

b. If a fulcrum is located at your estimate, determine the total number of weighted units to the right of the fulcrum.

c. Similarly, determine the total number of weighted units to the left of the fulcrum of your estimated point E.

d. Would you revise your estimate of point E in part **a** based on your answers to **b** and **c**?

8. Copy and complete the following chart for the diagram.

Point	Weight W_i	Position on number line x_i	Weighted value $W_i x_i$
A	______	______	______
B	______	______	______
C	______	______	______
D	______	______	______

9. Using the values determined from the chart in problem 8, calculate the location of the balance point by calculating the weighted mean. Locate and label this point as F on the diagram sketched in problem 7.

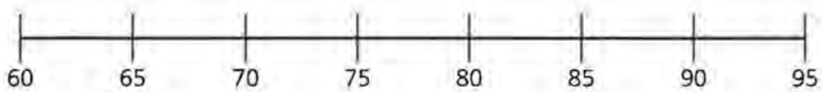
10. If the unit of weights were stated in grams instead of "the weight of a raisin," would the location of the balance point change? Why or why not?

Changing Units on the Number Line

In the previous problems, the number line was a convenient representation of the locations of weights. Many examples of weighted averages involve a similar setup, except that information other than location and weight is involved. Consider the following example.

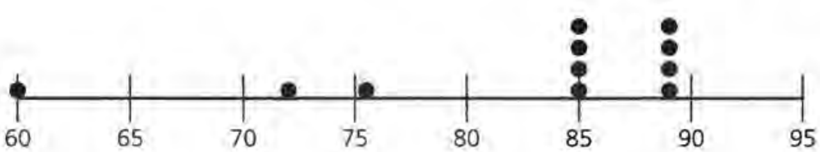
> Students enrolled in Ms. Clifford's math class frequently take either quizzes or exams. An exam is worth four times as much as a quiz. At the end of a grading period, Gail completed two exams and three quizzes. She got a score of 85% on the first exam, 89% on the second exam, and 72%, 60%, and 76% on the three quizzes. What is Gail's average for this grading period?

11. Before you begin to actually calculate Gail's average, consider a "picture" of this problem. A number line is sketched below. Copy it.



a. Instead of locations, what do the values indicated along this number line (60, 65, etc.) represent?

b. Consider locating ● along your number line. Instead of representing a unit of equal weight, what does a ● represent?



c. How is an exam represented in this picture?

d. What does the balance point of this diagram represent?

12. Copy and complete the following chart.

Exam or Quizzes	Number of units of weight W_i	Score on the number line x_i	Weighted value $W_i x_i$
1st Exam	4	85%	______
2nd Exam	______	______	______
1st Quiz	______	______	______
2nd Quiz	______	______	______
3rd Quiz	______	______	______

Calculate the weighted mean of the scores.

13. Steve is also in Ms. Clifford's class. He has not yet completed the third quiz. He scored 80% on the first exam, 84% on the second exam, 88% on the first quiz, and 80% on the second quiz. Steve is hoping a good performance on the third quiz can pull his average for this grading period up to 85%. Is this possible? Find the highest possible average Steve could receive after he completes the third quiz.

14. What is the lowest average Steve could receive for this grading period?

Practice and Applications

15. Consider again the problem from Lesson 2 involving the number of steps students recorded to walk a distance of 50 meters. The following data was organized from Ms. Clifford's class.

Number of students	Number of steps to walk 50 yards
3	55
2	60
5	62
4	64
1	65

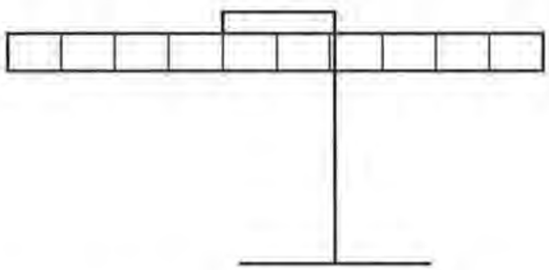
a. If everyone participated in this project, how many students are in Ms. Clifford's class?

b. Draw a number line with equal segments marked off to represent the units discussed in this problem. What units are represented on this number line?

c. Complete the drawing by placing ● on the number line as indicated by this problem.

d. Determine the average number of steps this class takes to walk the 50 meters.

16. An apparatus was developed in a physics lab that investigates the location of a balance point. A relatively lightweight rod was calibrated in inches, starting with 0 and ending with 30 inches. A weight of 15 grams was suspended from the 0 location of the rod, 20 grams at the location 15 inches, and 10 grams at the location of 30 inches.



a. Complete a copy of this picture by showing where the weights would be suspended.

b. What replaces the raisins in this picture?

c. If the weight of the rod is not considered, at what location along this rod would a balance of the suspended weights be obtained?

17. There are 26 students in Mr. Landwehr's Geometry class: 12 boys and 14 girls. An exam was completed by each of the students. The boys averaged 73.2% and the girls averaged 80.5%.

a. If a number line were constructed to represent this problem, what scale or units would be represented on this number line?

b. Represent this problem on a number line.

c. If the symbol ● was used in this problem, what does each ● represent?

d. What was the overall class average for this exam?

18. In Ms. Mastromatteo's Geometry class, 12 girls average 82.1% on this exam. The overall class average for 22 students was 84.4%. What was the average for the 10 boys?

Extension

19. Consider the following problem:

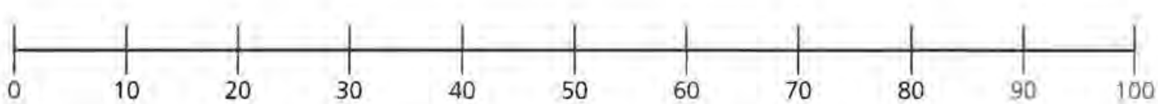
> On February 11, 1993, the Phoenix Suns played the Golden State Warriors in an NBA basketball game. Phoenix won the game by a score of 122 to 100, largely because one of the Phoenix players, Danny Ainge, made several three-point shots. Here is a list of those players involved in three-point shots at this game:

Player	3-Point Shots Attempted	3-Point Shots Completed	$\frac{\text{Completed}}{\text{Attempted}}$	% Completed
Barkley	1	0	$\frac{0}{1}$	0.0%
Majerle	7	1	$\frac{1}{7}$	approximately 14.3%
Ainge	12	7	$\frac{7}{12}$	approximately 58.3%
Team Totals	______	______	______	______

Complete the above chart given the data for the three players.

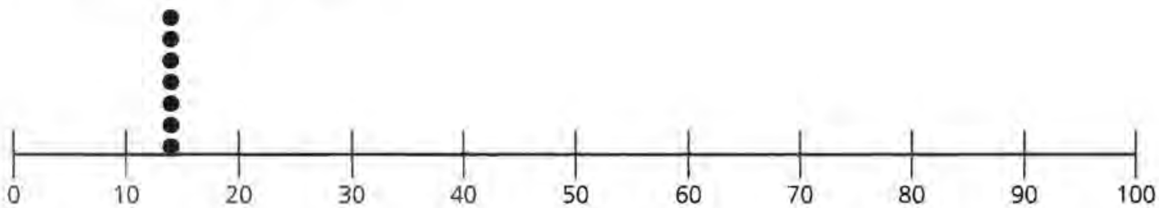
The team's three-point percentage could be found by various methods. Consider the following setup to determine the team's three-point percentage.

20. This number line was started to illustrate the team's three-point average.



What do the units (i.e., 0, 10, 20, etc.) represent on this number line?

21. Seven • are placed on this number line representing one of the players involved in this problem. Who is represented by these seven units and what does each unit represent? Why are the • symbols placed on the number line at a position between 10 and 20?



22. Copy and complete the above number line representing each of the players attempting a three-point shot for the Phoenix Suns.

23. Using the setup suggested by this diagram, find the team's average three-point shooting percentage.

24. Using the chart in problem 19, what was the team's average three-point shooting percentage?

25. Are the averages determined in problem 23 and problem 24 the same? Should they be?

"Raisin" Country

Imagine objects floating out in space but held together by some invisible wire. A point at which the objects will balance with support of a "fulcrum" represents an important center but one that is hard to visualize and even harder to model. What if the objects were all part of the same plane? Although this simplifies the example, the factors involved in finding a balance point begin to emerge. A two-dimensional example of objects on a plane can be modeled for purposes of this investigation. Raisins attached to a sheet of poster paper provide a reasonable model for estimating an important center.

What does this "center" indicate about the arrangement of the raisins? Does the center of three raisins arranged as a triangle indicate anything special about triangles? What if four raisins were arranged as a parallelogram or five raisins as a pentagon?

A balance point of objects arranged on a plane will be the primary investigation of this unit of *Exploring Centers.* As you study the shapes and special characteristics of the shapes in these lessons, develop a "map" of the general location of each student in your classroom. Where is the "center" of your class? Does this center indicate anything about your class? What might cause a teacher to attempt to change this center?

LESSON 5

Balancing a Point-Mass Triangle

Do you think there is a balance point for three points located on a plane?

If a balance point exists, how can you find it? Is there a model similar to the ruler, raisins, and pencil?

When might it be important to know this balance point?

Balancing equal weights on a number line involves finding a point or center that evenly distributes the total distances of each weight to the left and to the right of the balance point. Expand this idea to three noncollinear points. Consider objects of equal weight located at three points on a sheet of paper.

OBJECTIVES

Determine the balance point of a point-mass triangle by experimentation.

Determine the centroid of a point-mass triangle.

Construct the intersection of the medians of a triangle.

Summarize the relationship of balance point, centroid, and intersection of medians.

INVESTIGATE

Balancing the Triangle

Triangles represent one of the most basic geometric shapes you will study. More complex geometric shapes are frequently investigated by dividing them into triangles! Triangles are described and classified by their angle measures and the lengths of their sides.

- A triangle in which each of its three angles has a measure less than 90 degrees is called an *acute triangle.*
- A triangle in which each all three sides are of equal length is called an ***equilateral triangle.***

Discussion and Practice

1. Research a geometry book and write a short summary of the angle descriptions and side descriptions for the following types of triangles.

Angle Descriptions	Side Descriptions
Acute Triangle: A triangle in which each of the three angles has a measure less than 90 degrees.	**Scalene Triangle:**
Obtuse Triangle:	**Isosceles Triangle:**
Right Triangle:	**Equilateral Triangle:**

2. Sketch a diagram of a triangle classified as an ***isosceles right triangle.***

3. Look at the triangles on *Activity Sheet 4*. Select Triangle 1 and one other triangle. Determine the following measures of the triangles by using a protractor and ruler if necessary.

Triangle 1	Triangle ______ (your choice)
Length of P_1P_2: ______	Length of P_1P_2: ______
Length of P_2P_3: ______	Length of P_2P_3: ______
Length of P_1P_3: ______	Length of P_1P_3: ______
Measure of angle P_1: ______	Measure of angle P_1: ______
Measure of angle P_2: ______	Measure of angle P_2: ______
Measure of angle P_3: ______	Measure of angle P_3: ______

4. Refer to the solutions to problem 3.
   a. Was it necessary to measure each angle with the protractor? Why or why not?
   b. Are there any special characteristics of a triangle you used to determine the value of an angle or a side? If yes, describe the characteristics.

5. Using the approximate angle and side values summarized in problem 1, how would you describe:
   a. Triangle 1
   b. Triangle ______ (your choice)

6. In this investigation, you will use poster paper, raisins, and a pencil to investigate the balance point of a triangle. The triangle is outlined by poster paper. Raisins will be placed as objects of approximately equal weight at each vertex. Balancing this model finds a location that balances the weight and distribution of the poster paper and the raisins. The raisins represent a model called a ***point-mass*** distribution. The poster paper represents a model called a ***lamina.***

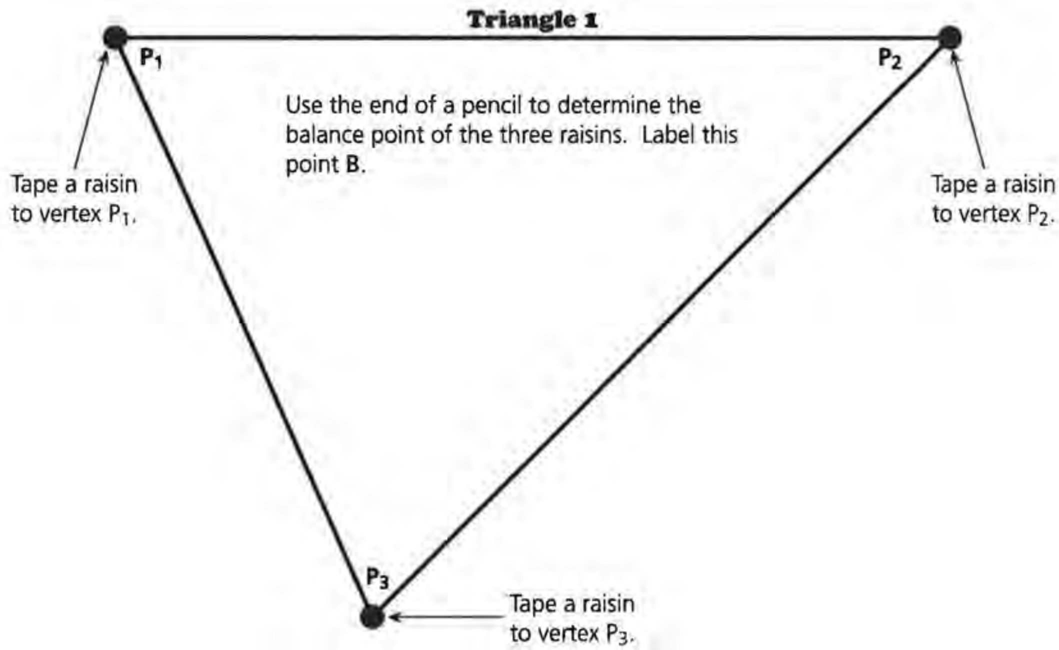
   a. Cut out Triangle 1 from *Activity Sheet 4.*

   b. Trace the shape of this triangle on a firm piece of poster paper, labeling each vertex of the triangle as indicated from the diagram. Cut out the triangle from the poster board.

   c. Tape one raisin at each vertex of the triangle (and as close to the vertex as possible).

   d. Balance this triangle on a pencil. You may want to use the eraser end of a new pencil. Warning! Don't give up too easily if the shape does not balance. Finding a balance point requires adjusting the pencil position in small increments. Keep in mind you are using the pencil as a fulcrum to locate a balance point similar to the point on a number line or ruler in the earlier lessons. (Suggestion: attempt to balance the figure with the raisins facing down. This helps stabilize the figure.)

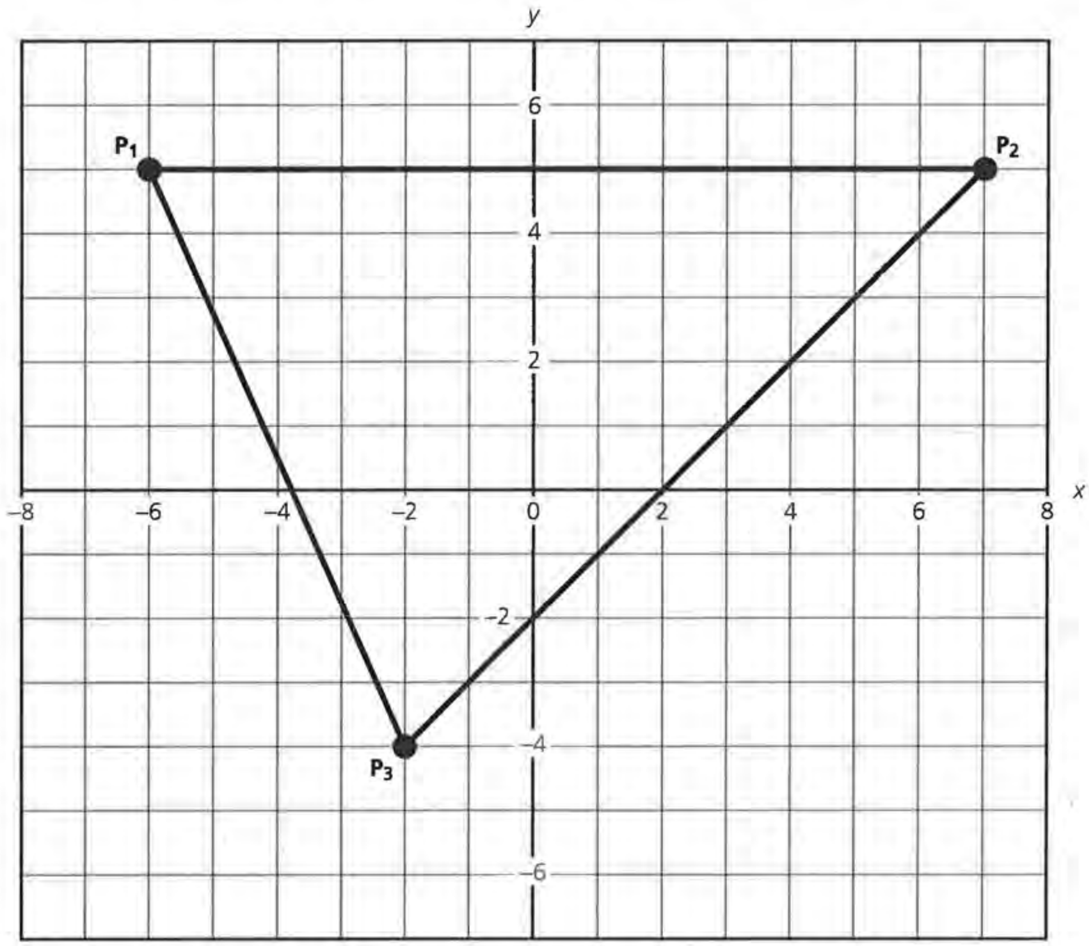
7. When you are able to balance the figure, mark the position on the triangle as point **B** for "balance point." See the following diagram.



The balance point **B** is the point that "centers" the distribution of the weights and locations of the raisins and the poster paper. The weight of the poster paper is uniformly distributed in this model. As a result, point **B** is primarily determined by the locations and weights of the raisins.

Calculating the Centroid

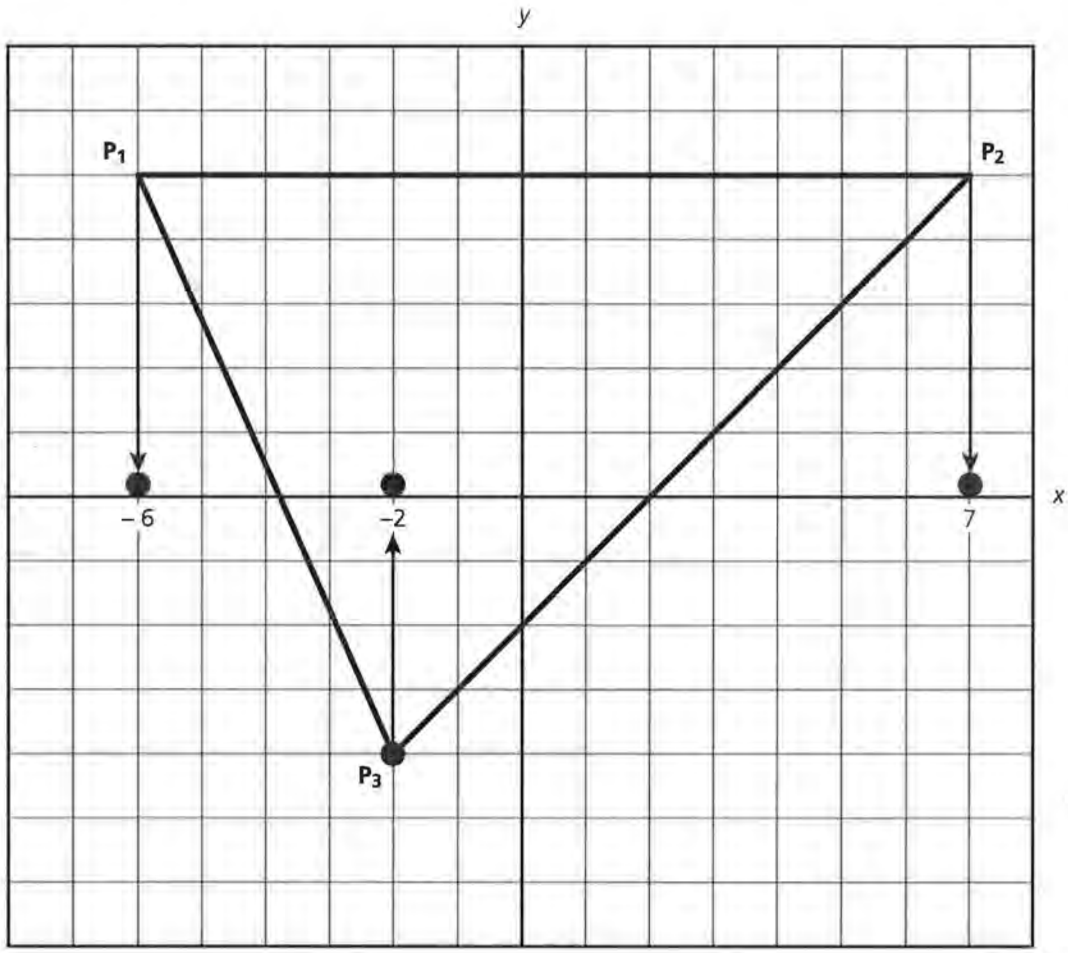
If you assume the weights of the raisins are equal, there is another method to determine **B**, the balance point. It is found by placing the triangle on an *xy*-coordinate system.



8. Estimate the coordinates of point **B** from the above placement of Triangle 1. Complete the following table based on this particular placement:

Point	*x*-value	*y*-value
P_1	______	______
P_2	______	______
P_3	______	______
B	______	______

The balance point for this figure requires calculating both an x-value and a y-value. The horizontal component (or x-value) of this point can be found by imagining the raisins along the x-axis.

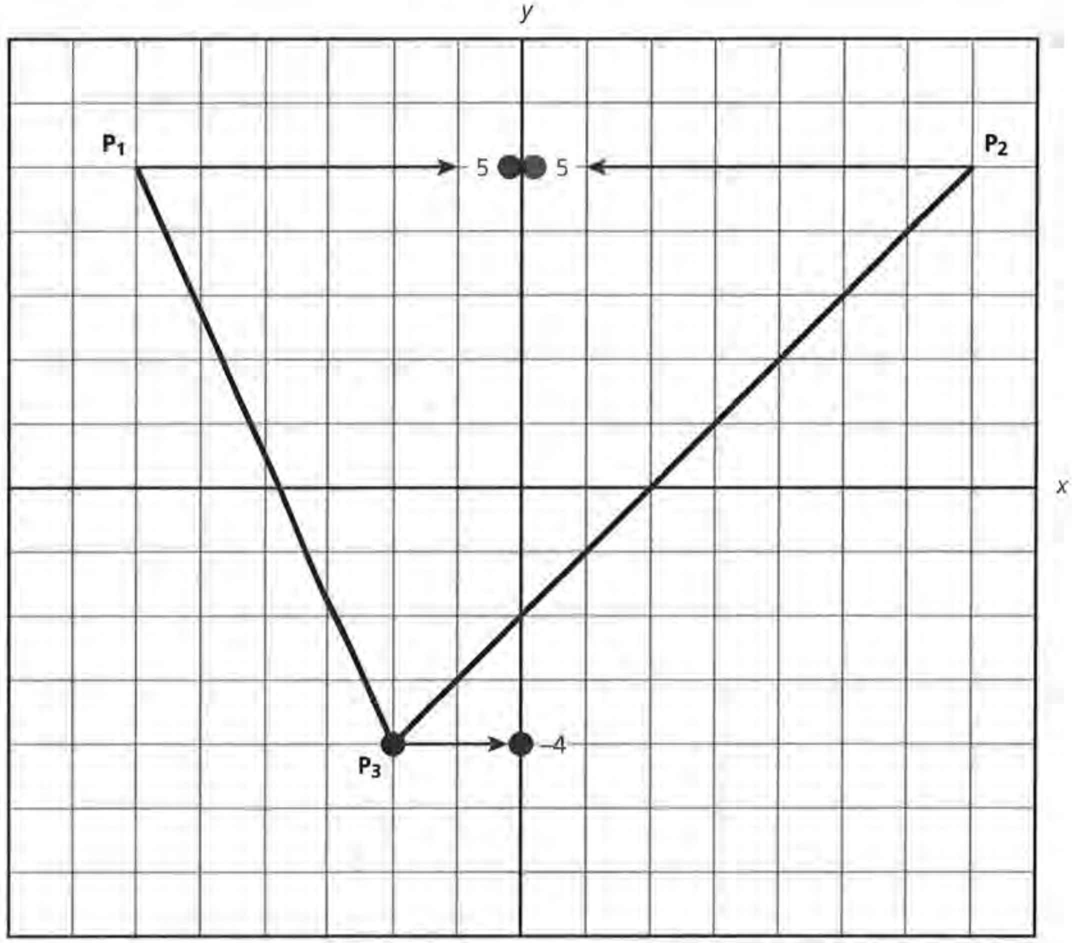


Recall from your work in Lesson 3 that the mean of points on a number line determined the balance point for weights of "equal" value. This was summarized as:

$$\bar{x} = \frac{(x_1 + x_2 + x_3)}{3}$$

9. Find the value of $\bar{x}$ using Triangle 1 as pictured above.

A similar approach can be taken to determine the vertical component (y-value) of the balance point.



The y-value of the balance point would be determined by the mean of the positions of the raisins along the y-axis. This is summarized as:

$$\bar{y} = \frac{(y_1 + y_2 + y_3)}{3}$$

10. Determine the value of $\bar{y}$ using Triangle 1 as pictured above.

11. The balance point calculated in this way is called the ***centroid.*** It is represented by C ($\bar{x}$, $\bar{y}$). Use the values calculated in problems 9 and 10 to locate the centroid of this particular triangle. (Label this point as **C**.)

12. How does the location of C compare to your location of **B**?

13. The balance point **B** is an approximation of the centroid C. What could cause the balance point obtained by experimentation to be different than the centroid?

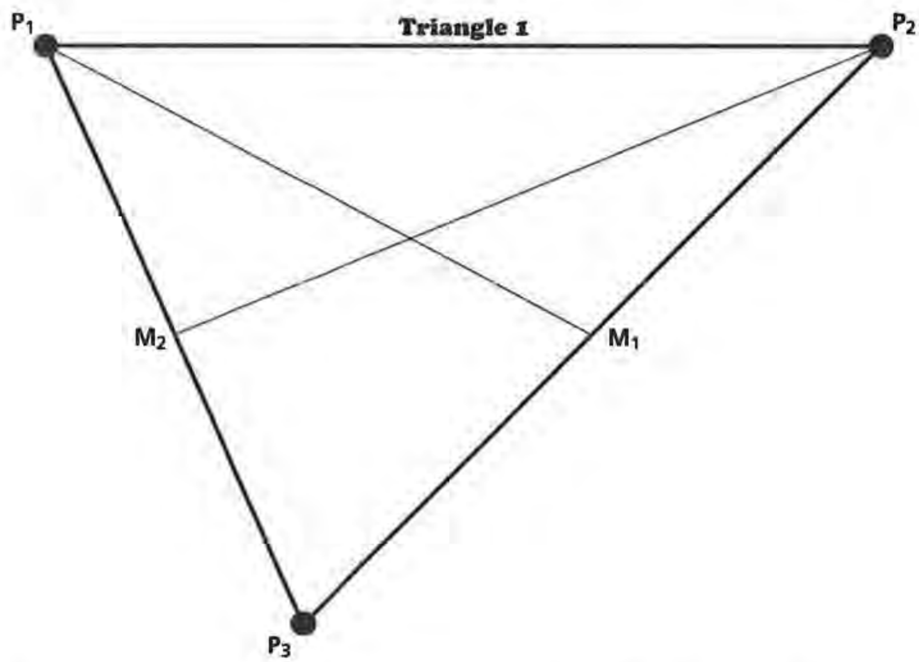
14. The centroid "balances" the vertical and horizontal distances of the raisins on the plane. Based on the placement of Triangle 1 in the previous diagrams, determine the following:

a. $(x_1 - \bar{x}) + (x_2 - \bar{x}) + (x_3 - \bar{x}) =$

b. $(y_1 - \bar{y}) + (y_2 - \bar{y}) + (y_3 - \bar{y}) =$

Working with the Medians

Several well-known and important theorems in geometry point out the special features of the ***medians*** of a triangle. A median of a triangle is a line segment that connects a vertex to the ***midpoint*** of the side opposite that vertex. In **Triangle 1,** $\overline{P_1M_1}$ and $\overline{P_2M_2}$ are examples of medians:



An important theorem in geometry states ***"The three medians of a triangle intersect at one point."*** If you have not previously studied this theorem, you might want to experiment by constructing the three medians for each of the models on *Activity Sheet 4*. Make certain you verify the accuracy of this important theorem.

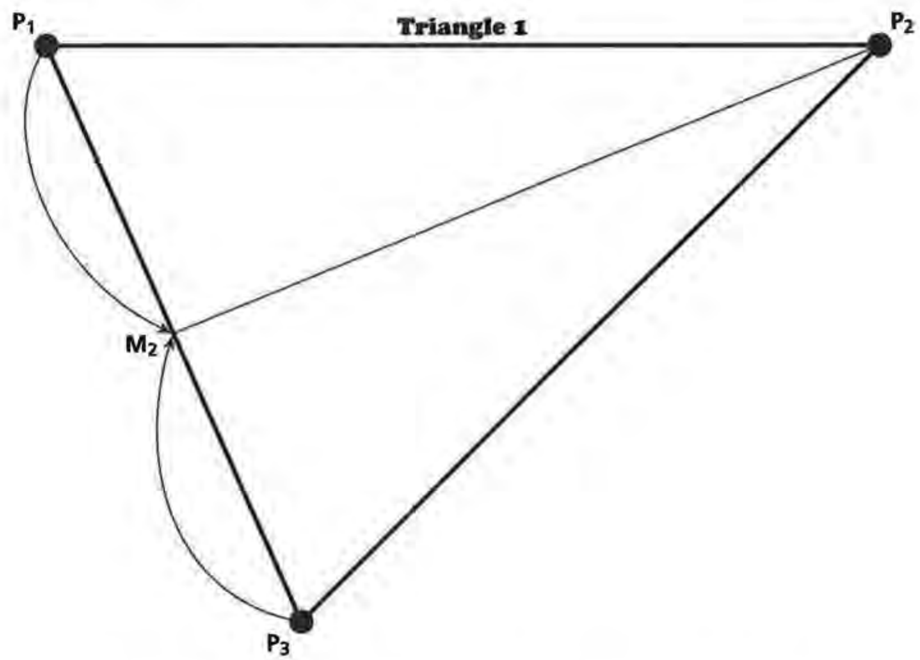
15. Return to the triangle you cut out of the poster board. Carefully measure the length of each side to determine its midpoint. Label midpoints as M_1, M_2, and M_3. Connect each midpoint to the opposite vertex.

a. Do the three medians intersect at a point? If yes, label this point as M. (If your medians do not intersect, carefully remeasure the midpoints of the triangle. Remember the theorem!)

b. Compare M to locations of centroid C and the balance point B. Are the locations similar?

c. Complete the following statement based on the locations of M, C, and B: "The medians of a triangle intersect at ________________."

Another important summary can be illustrated with the original arrangement of raisins taped to P_1, P_2, and P_3 of **Triangle 1**. Consider $\overline{P_1P_3}$ of the triangle formed with one raisin located at vertex P_1 and one raisin located at vertex P_3. (Again, consider each raisin to be of equal weight.)



The two raisins taped to the endpoints of this segment can be moved to M_2 or the midpoint of $\overline{P_1P_3}$. This is called ***collapsing the raisins.***

Remove the raisins from P_1 and P_3. Combine the two raisins at location M_2. The balance point B of the original triangle is also the balance point of the raisins arranged along segment $\overline{M_2P_2}$.

16. Moving the two raisins to M_2 changes the arrangement of the three raisins but not the balance point B.

a. Why do you think B remains unchanged?

b. Review the following formulas for determining the midpoint:

$$\frac{x_1 + x_2}{2} \text{ and } \frac{y_1 + y_2}{2}$$

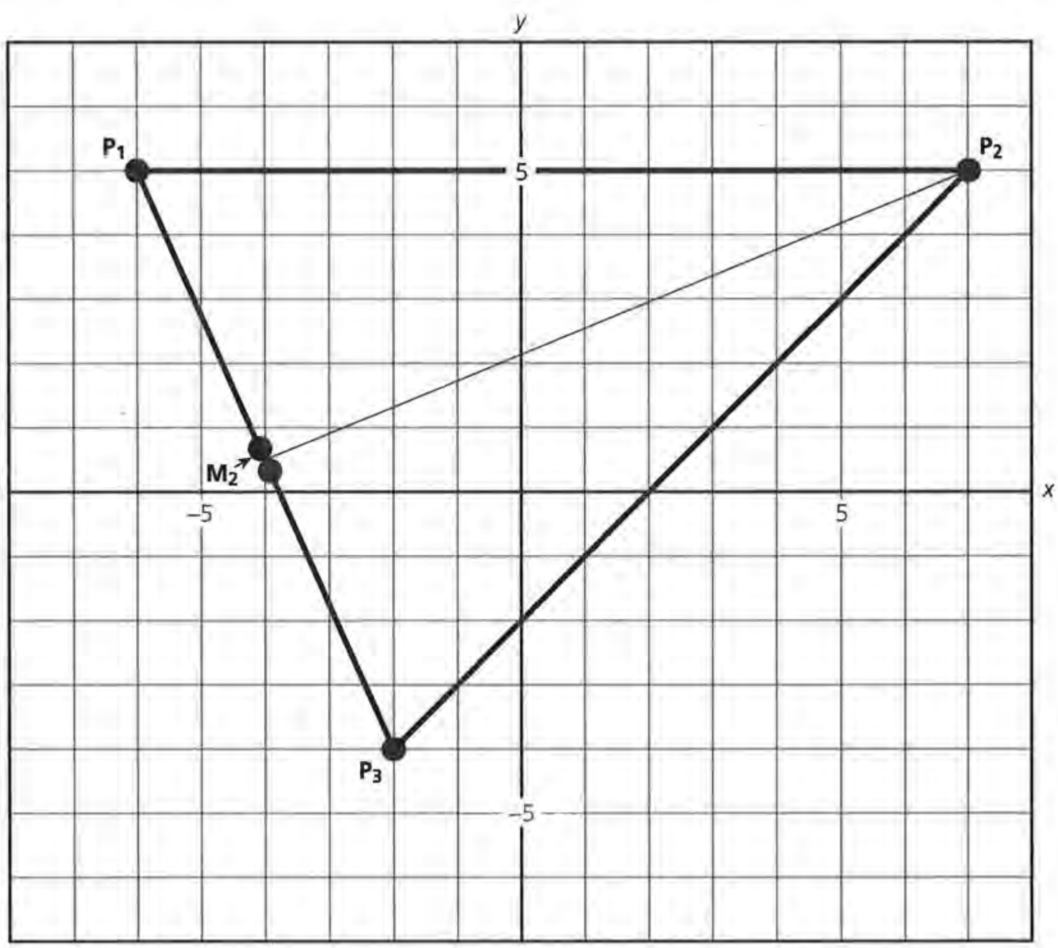
As M_2 is the midpoint of P_1 and P_3, the x-coordinate value of M_2 is:

$$\frac{-6 + -2}{2} = \frac{-8}{2} = -4$$

and the y-coordinate value of M_2 is:

$$\frac{5 + -4}{2} = \frac{1}{2} = 0.5$$

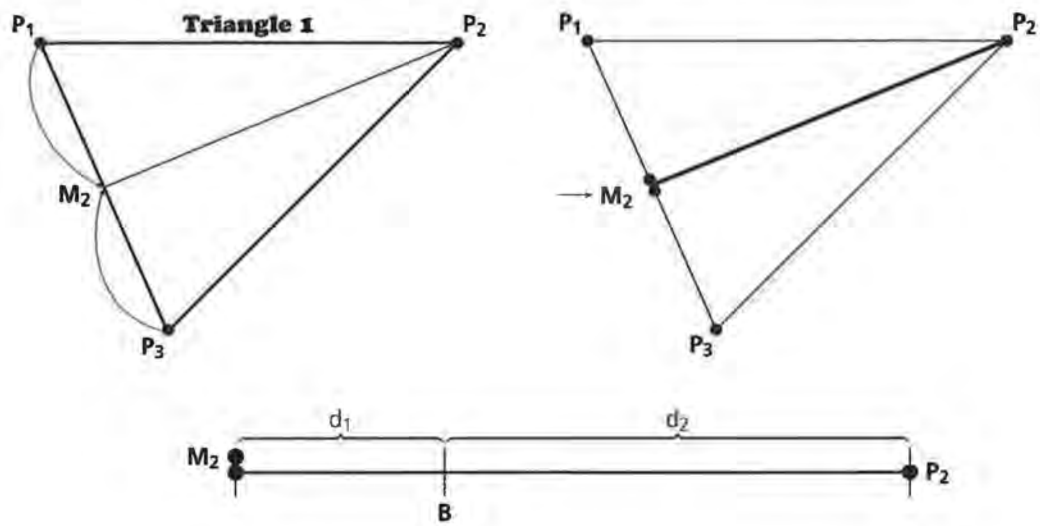
This location of M_2 can be verified by the previous placement of the triangle in the coordinate grid. Assume each tick mark is one unit.



Determine $\bar{x}$ based on the placement of 2 raisins at $\mathbf{M_2}$ and 1 raisin at $\mathbf{P_2}$. Similarly, determine $\bar{y}$.

Is this balance point $(\bar{x}, \bar{y})$ the same as point **B**?

Return to the rearrangement of raisins by collapsing the 2 raisins to $\mathbf{M_2}$ as illustrated in the following diagram:



Examine your triangle. Point **B**, discovered earlier in this lesson, should also be part of the segment $\overline{P_2M_2}$. Explain.

There exists a special relationship between d_1 and d_2 as illustrated above. Recall from Lesson 4 that the total distances for each unit of weight (or raisin) on one side of the balance point equals the total distances for each unit of weight on the other side. In this example, the resulting balance contributed by the two raisins at point $\mathbf{M_2}$ and the one raisin at point $\mathbf{P_2}$ is:

$$2d_1 = 1d_2$$

Also observe that:

$$d_1 + d_2 = M_2P_2$$

Therefore, by substitution,

$$d_1 + 2d_1 = M_2P_2$$

$$3d_1 = M_2P_2$$

$$d_1 = \frac{1}{3} M_2P_2 \text{ and } d_2 = \frac{2}{3} M_2P_2$$

17. The above work indicates the location of the balance point is $\frac{1}{3}$ of the distance from $\mathbf{M_2}$ and $\frac{2}{3}$ of the distance from $\mathbf{P_2}$. What does this indicate about the intersection of the medians **M** or the location of the triangle's centroid **C**?

18. Suppose that instead of collapsing the raisins at $\mathbf{P_1}$ and $\mathbf{P_2}$ you collapsed the raisins located at vertex $\mathbf{P_2}$ and at vertex $\mathbf{P_3}$ to the midpoint $\mathbf{M_1}$. Then $\overline{P_1M_1}$ would also be a median of the triangle. Copy the following representation of segment $\overline{P_1M_1}$ and the attached raisins.

a. Estimate on your copy of this segment the location of the **balance point** of $\overline{P_1M_1}$.

b. How is the balance point you estimated along $\overline{P_1M_1}$ related to the balance point **B** of the triangle?

c. When you collapse raisins, you are changing the arrangement of the raisins or weights but you are not changing the location of the balance point. Why is **B** (or **C**) also part of $\overline{P_1M_1}$?

19. According the theorem mentioned in this lesson, the three medians of a triangle intersect in one point. If the location of the intersection of the medians was based on the location you estimated for **B**, how would you explain that this point is also the intersection of the medians?

SUMMARY

Three raisins taped to a sheet of poster paper form a model of a point-mass triangle. The location of the **centroid** or point that balances the raisins can be estimated by:

a. using a pencil as a fulcrum to locate a point that balances the model

b. determining the means of the x- and y-coordinates of each of the raisins

c. locating the intersection of the medians

Practice and Applications

20. Cut out the second triangle you selected from the *Activity Sheet 4* on poster paper. Tape a raisin at each vertex. Using a pencil, find the point that balances the triangle. Label this as point B on your triangle.

21. Place your cut-out triangle on a coordinate system. Determine the x- and y-coordinates for each vertex and the balance point B. Copy and complete the following table for the triangle you selected:

Point	*x*-value	*y*-value
P_1	______	______
P_2	______	______
P_3	______	______
B	______	______

22. Determine $\bar{x}$ and $\bar{y}$ for the vertices.

23. Locate $(\bar{x}, \bar{y})$ as point C on the cut-out form of your triangle. (This represents the centroid of the triangle.)

24. Consider the following:

- **a.** If the point that balances the triangle with your pencil is the centroid for a point-mass triangle, what would you expect about the points B and C?
- **b.** Why might the specific locations of points B and C not turn out as expected?

25. Measure and mark the midpoints of each of the sides of your triangle. Label the midpoint of $\overline{P_2P_3}$ as M_1, $\overline{P_1P_3}$ as M_2, and $\overline{P_1P_2}$ as M_3. Connect each midpoint to the opposite vertex of the triangle. Do the medians you constructed intersect at one point? If yes, mark the point of intersection as M.

26. Using your triangle, measure the listed segments with a ruler and calculate the ratios:

$P_1M =$	$MM_1 =$	$\dfrac{P_1M}{MM_1} =$
$P_2M =$	$MM_2 =$	$\dfrac{P_2M}{MM_2} =$
$P_3M =$	$MM_3 =$	$\dfrac{P_3M}{MM_3} =$

27. Summarize the ratios found in the table above.

LESSON 6

Investigating Quadrilaterals

Which do you think is more stable, a 3-legged stool or a 4-legged stool? Why?

Imagine a quadrilateral with a raisin taped to each of the vertices. Assume each raisin weighs the same. Do you think there is one point on the plane of the quadrilateral designating the location of a fulcrum that would balance the four raisins? Why or why not?

OBJECTIVES

Determine the balance point of objects forming a quadrilateral on a plane through experimentation and coordinate geometry.

Identify special characteristics of a parallelogram by locating the balance point of a point-mass model.

Describe concavity by the location of a balance point.

Estimate the balance point of a point-mass model by collapsing raisins.

An extension of the investigation involving triangles is to attach additional objects of equal weight to the plane. How does a fourth object on the plane change the location of a balance point of the weighted objects?

INVESTIGATE

Parallelograms

Locating a fourth object on the plane outlines another familiar shape called a ***quadrilateral.*** A quadrilateral is defined as a four-sided, closed-plane figure. Squares, rectangles, parallelograms, and trapezoids are a few special subsets of the larger set of quadrilaterals.

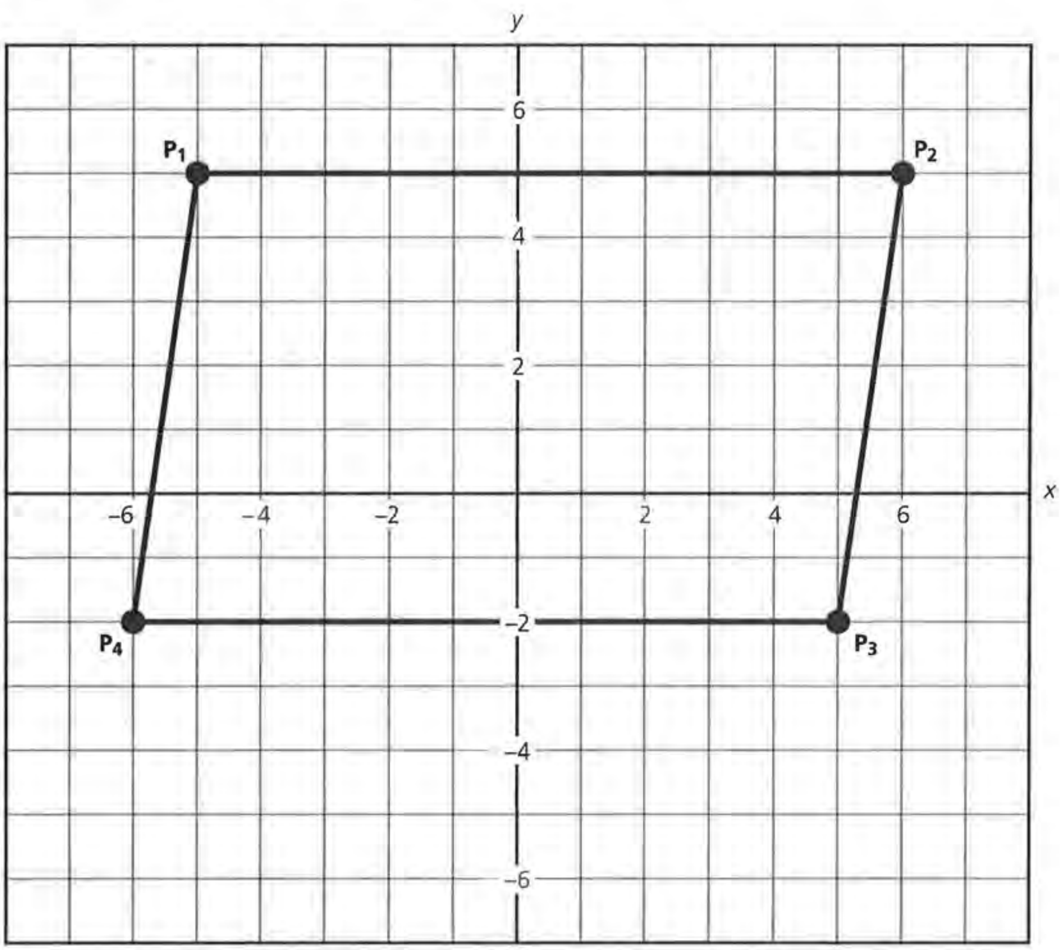
A good starting point is to examine a ***parallelogram.*** Use Quadrilateral 1 from *Activity Sheet 5* to investigate the following problems.

Discussion and Practice

1. Quadrilateral 1 is a parallelogram. Using a protractor and a ruler, find and record the measures of the sides and angles of Quadrilateral 1.

Measure of Angle		Length	
P_1 =	______	P_1P_2 =	______
P_2 =	______	P_2P_3 =	______
P_3 =	______	P_3P_4 =	______
P_4 =	______	P_4P_1 =	______

2. Examine your recorded measurements for Quadrilateral 1.
   a. Describe at least three special characteristics of this parallelogram.
   b. Which characteristics do you think are true of all parallelograms?
3. Tape raisins to each vertex of the quadrilateral.



   a. Using a pencil as a fulcrum, estimate the location of the balance point of the four raisins. Mark and label your estimate on the cut-out figure as point **B**.

b. Was B where you expected it to be located? Why or why not?

Recall that the calculation of the centroid C ($\bar{x}$, $\bar{y}$) could be used to locate the balance point of a triangle; $\bar{x}$ represents the mean of the x values of the vertices and $\bar{y}$ represents the mean of the y values of the vertices.

4. Copy and complete the following table based on the placement of **Quadrilateral 1** in the xy-coordinate grid as previously illustrated.

Point	x-value	y-value
P_1	______	______
P_2	______	______
P_3	______	______
P_4	______	______

a. Determine the value of centroid C ($\bar{x}$, $\bar{y}$).

b. Estimate the coordinate values of B. Was your estimate of the balance point using the pencil close to the coordinate location of the centroid?

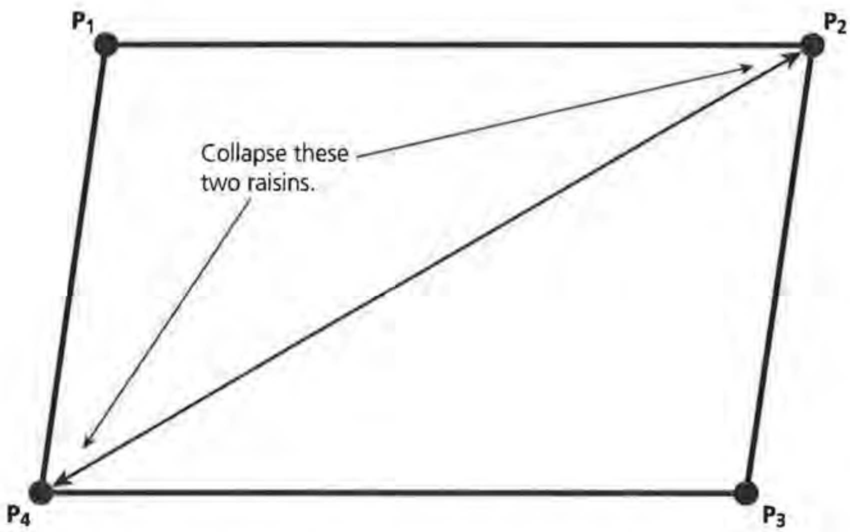
c. Determine the approximate number of centimeters separating your estimate and the calculated location of the centroid C.

d. The centroid of a triangle is located at the intersection of the medians. Do you think the intersection of the medians would locate the centroid for a parallelogram? Why or why not?

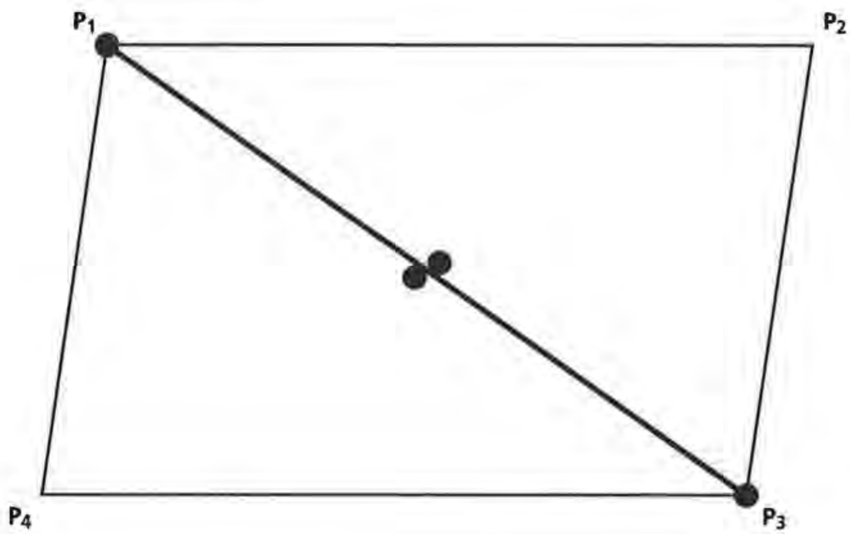
A geometry theorem states, "**The diagonals of a parallelogram bisect each other.**" If you have not previously studied this theorem, take some time to work with the parallelogram cut out. In addition to the measurements recorded earlier, draw and measure the segments representing the diagonals. Measure the distances from the intersection of the diagonals to each of the vertices. Keep this theorem in mind as you consider the balance point of four raisins outlining a parallelogram.

Your previous work demonstrated that the midpoint of the segment joining two objects of equal weight represents the balance point. Moving two raisins (or objects) to a position that preserves their balance and combines the weight of the raisins (objects) is called ***collapsing the raisins.***

5. Consider raisins located at vertex P_2 and vertex P_4. Also consider the segment connecting these two raisins. Collapse the raisins to produce a new arrangement that continues to balance at B. Describe where the raisins would be located.



6. A sketch of the new arrangement of the four raisins is illustrated below.



a. Notice the parallelogram is "collapsed" to a segment. Explain why the four raisins would be located along segment $\overline{P_1P_3}$.

b. At what position along this segment would the balance point be located for all four raisins? Why?

7. How does the balance point of the arrangement illustrated in problem 6 relate to the special characteristics of a parallelogram?

8. If four objects of equal weight outline a parallelogram in a plane, describe two ways you could locate the balance point or centroid.

9. There is another way to determine this balance point. Return to the original arrangement of one raisin located at each vertex P_1, P_2, P_3, and P_4. Develop sketches of the following rearrangements of the four raisins:

   a. Collapse the raisins located at P_1 and P_4.

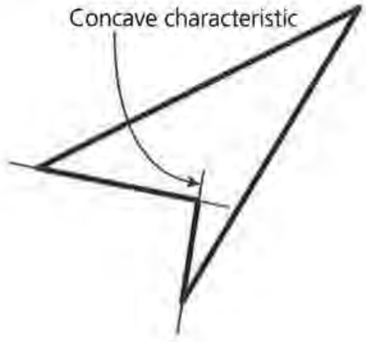
   b. Collapse the raisins located at P_2 and P_3.

10. According to your last sketch, two raisins are located at the midpoint of $\overline{P_1P_4}$ and two raisins at the midpoint of $\overline{P_2P_3}$. The midpoint of the segment connecting the two piles of raisins represents the balance point of the figure. Label this point on your sketch.

11. The balance point found in problem 10 represents the same point described in problem 6. What does this indicate about the segment joining the midpoints of the opposite sides of a parallelogram?

Concave Quadrilaterals

Quadrilateral 3 from *Activity Sheet 5* illustrates a different characteristic from the other quadrilaterals. This example is called a ***concave quadrilateral. Concave*** describes any shape with the characteristic that if you extend at least one side of the figure, the extended line would intersect in the interior region of the shape. ***Convex*** describes a shape in which extensions of a side would not intersect in the interior of the shape. Each of the other quadrilaterals included in the options sheet are convex.



12. Cut out **Quadrilateral 3** from *Activity Sheet 5* on poster paper. Tape a raisin at each of the vertices $\mathbf{P_1}$, $\mathbf{P_2}$, $\mathbf{P_3}$, and $\mathbf{P_4}$. Using the blunt end of the pencil, attempt to balance the figure. Describe any problems you encounter.

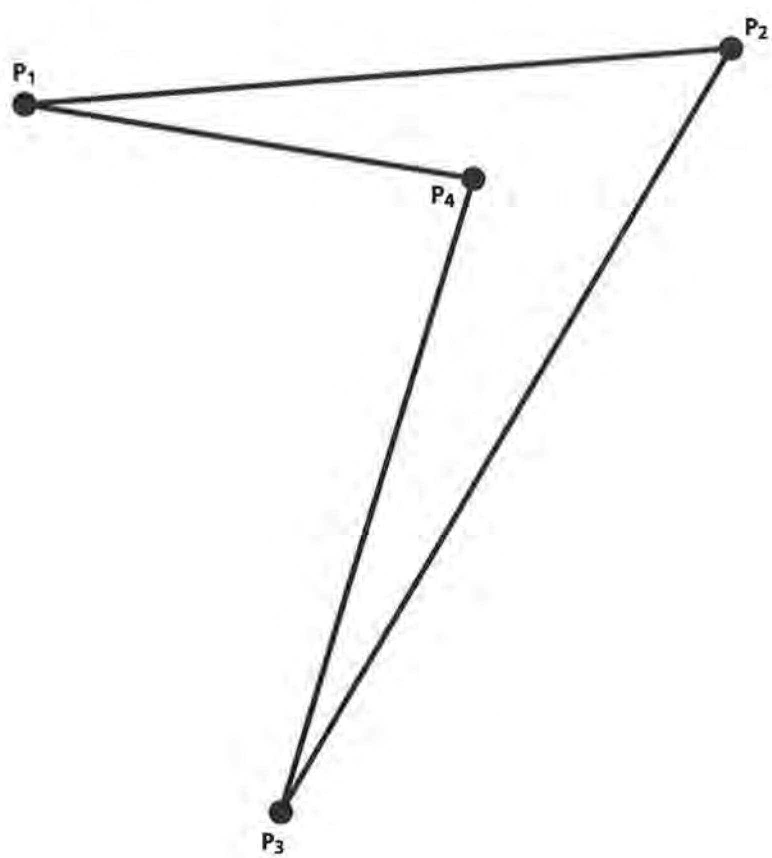
13. Place **Quadrilateral 3** on a coordinate grid. Record the coordinate values of each vertex. Copy and complete the following table for this shape.

Point	***x*-value**	***y*-value**
P_1	______	______
P_2	______	______
P_3	______	______
P_4	______	______

14. Using the values recorded on the table, determine the centroid or $\mathbf{C}(\bar{x}, \bar{y})$.

$$\bar{x} = \frac{x_1 + x_2 + x_3 + x_4}{4} \quad \text{and} \quad \bar{y} = \frac{y_1 + y_2 + y_3 + y_4}{4}$$

15. Consider the following steps in locating the balance point of the ***boomerang***. Develop a sketch of the rearrangements of the raisins for each step.



a. Collapse the raisins located at P_1 and P_4.

b. Collapse the raisins located at P_2 and P_3.

c. Two raisins are located at the midpoint of P_1P_4 and two raisins at the midpoint of P_2P_3. Sketch an estimate of the balance point.

d. Why did you have a problem balancing the original figure with four raisins?

16. Return the raisins to the points outlining the boomerang. Develop a sketch of the following steps:

a. Collapse the raisins located at P_1 and P_2.

b. Collapse the raisins located at P_3 and P_4.

c. Estimate the balance point.

d. Is this the same location you discovered in problem 15?

e. Summarize how to determine the balance point of a ***boomerang.***

SUMMARY

The balance point of four objects (of equal weight) arranged as a quadrilateral can be determined by experimentation and by calculation of the centroid. The balance point of four objects outlining a parallelogram is located at the intersection of the diagonals due to the special characteristics of a parallelogram. Objects of equal weight outlining a quadrilateral have a balance point located at the intersection of the segments connecting the two midpoints of opposite sides. This process works for all quadrilaterals.

Practice and Applications

17. Consider *Quadrilateral 2* (the ***Isosceles Trapezoid***).

a. Place a cut-out copy of the quadrilateral on a coordinate grid. Copy and record the following information:

Point	*x*-value	*y*-value
P_1	______	______
P_2	______	______
P_3	______	______
P_4	______	______

b. Why is this figure called an isosceles trapezoid?

c. Describe two methods to determine the balance point of four raisins outlining an isosceles trapezoid.

d. Determine the balance point of this quadrilateral using one of the methods described above.

e. Does the point representing the intersection of the diagonals help you locate the balance point for this quadrilateral? Why or why not?

18. Consider **Quadrilateral** 4 (the ***"Nothing Special" Quadrilateral***).

a. Is this quadrilateral convex? If yes, how do you know?

b. Describe two methods you could develop to determine the balance point of four raisins located at the vertices of this quadrilateral.

19. A theorem involving quadrilateral states:

"The figure formed by connecting the midpoints of each side of a quadrilateral is a parallelogram."

a. Sketch an outline of **Quadrilateral 4.** On this sketch, show the shape formed by attaching a raisin to the midpoint of each segment.

b. Determine the balance point of the four raisins positioned at the midpoints of the sides of **Quadrilateral 4.** How did you determine this balance point?

c. Is the balance point determined in part b the balance point of the four raisins outlining the pattern of **Quadrilateral 4?** Explain your answer.

20. Develop a sketch of **Quadrilateral 3.** Estimate the location of the balance point by the steps outlined in problem 19.

LESSON 7

Polygons!

Two flat tabletops are raised off the ground and placed on bricks for support. Would you consider a tabletop with six support bricks more stable than a tabletop with three support bricks?

If one of the support bricks is removed from each tabletop and you are asked to stand on one of the tabletops, which tabletop would you prefer? Why?

OBJECTIVES

Determine the balance point of a point-mass model forming a polygon by "collapsing" the objects.

Connect the method of collapsing the objects to the method of balancing the model using a fulcrum and to the method of calculating the centroid.

Generalize the methods of finding the balance point of the point-mass model.

Completing a study of four objects taped to the poster paper suggests extending an investigation to five or more objects. What happens to the location of a balance point as more objects are taped to the poster paper?

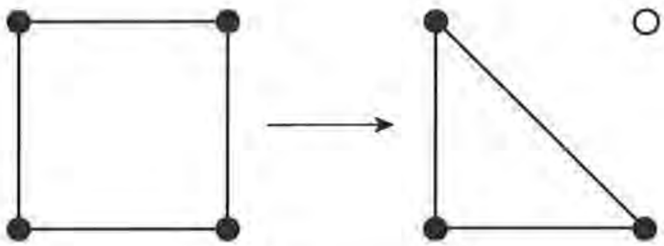
INVESTIGATE

Balancing Multiple Objects

Previous lessons indicated the weight of the raisins, the specific distribution of the raisins, and, to some extent, the weight of the poster paper affected the location of the balance point. How can an estimated balance point be determined for any number of objects taped to a plane?

Discussion and Practice

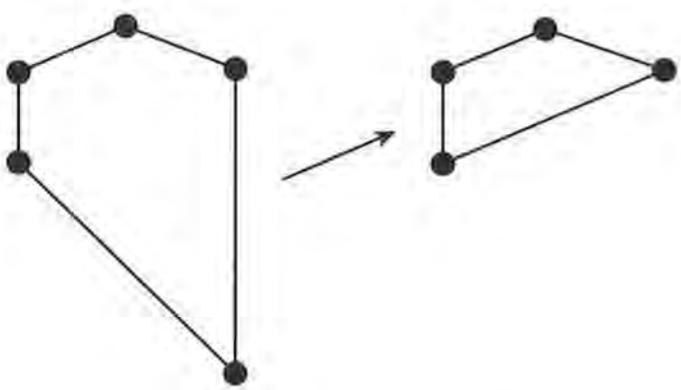
1. Four raisins were taped to poster paper as indicated in the first of the following two diagrams. One raisin was removed and the shape recut on the poster paper as indicated. Estimate the change in the balance point from removing one raisin by estimating the balance point for each arrangement.



2. Design an arrangement of four raisins on the poster paper that would result in a more noticeable shift in the balance point by removing one raisin and recutting the paper. Indicate in your design which raisin you remove to produce the shift in balance.

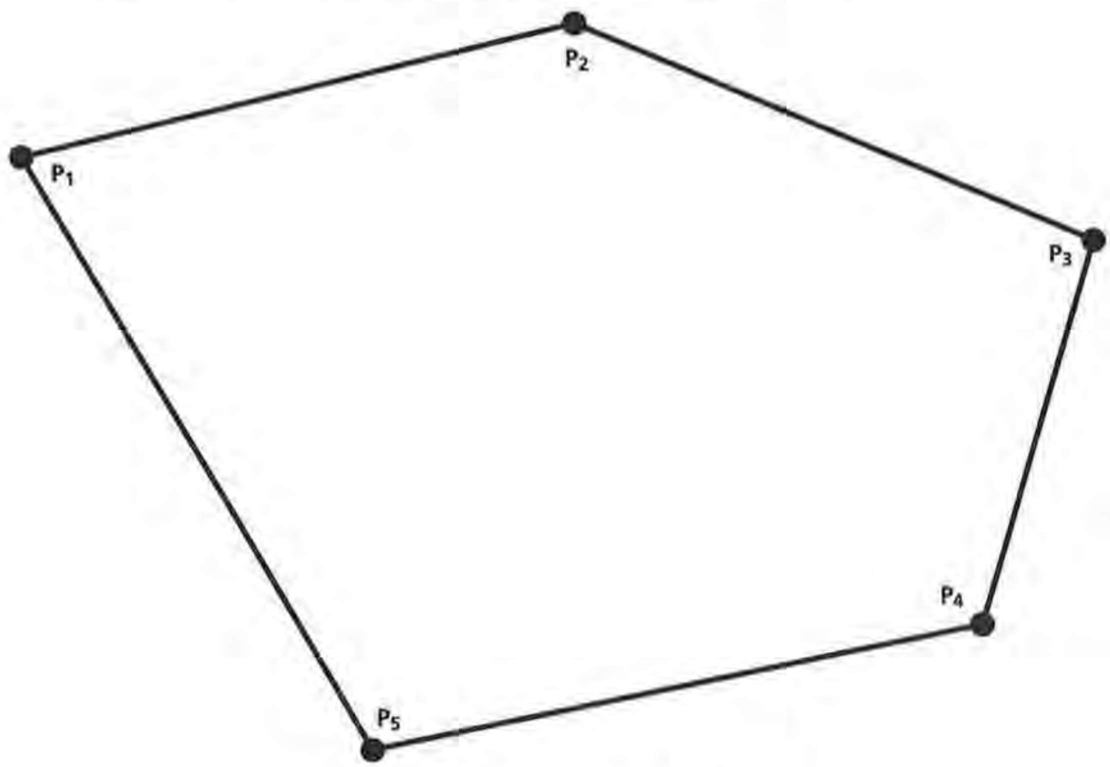
The balance point is influenced by the weight of the poster paper, the weight of the raisins, and the distribution of the raisins. The paper represents a uniform distribution of weight (an example of a ***lamina***). Given the type of raisin models investigated in this module, the poster paper has a minor contribution in estimating the balance point. The primary factors in estimating a balance point in the models are the locations and weights of the raisins. As indicated in the previous lessons, this part of the model is called a ***point-mass distribution.***

3. A model of five raisins is illustrated in the following drawing. A raisin is removed and the new shape recut. Indicate if you think the shift in the resulting balance point of the objects would be minor or rather noticeable. Explain your answer.



4. What methods could you use to find the balance point of a point-mass distribution of five or more objects placed on a plane?

The balance point of five objects of equal weight (raisins) outlining a convex polygon will be investigated in this lesson. A good starting point is the following pentagon (five-sided polygon):



The vertices of this polygon are identified as P_1, P_2, P_3, P_4, and P_5. Trace this figure on another sheet of paper and copy it to poster paper. Make sure you record the labels of the vertices on the poster paper, and then cut out the polygon from the poster paper.

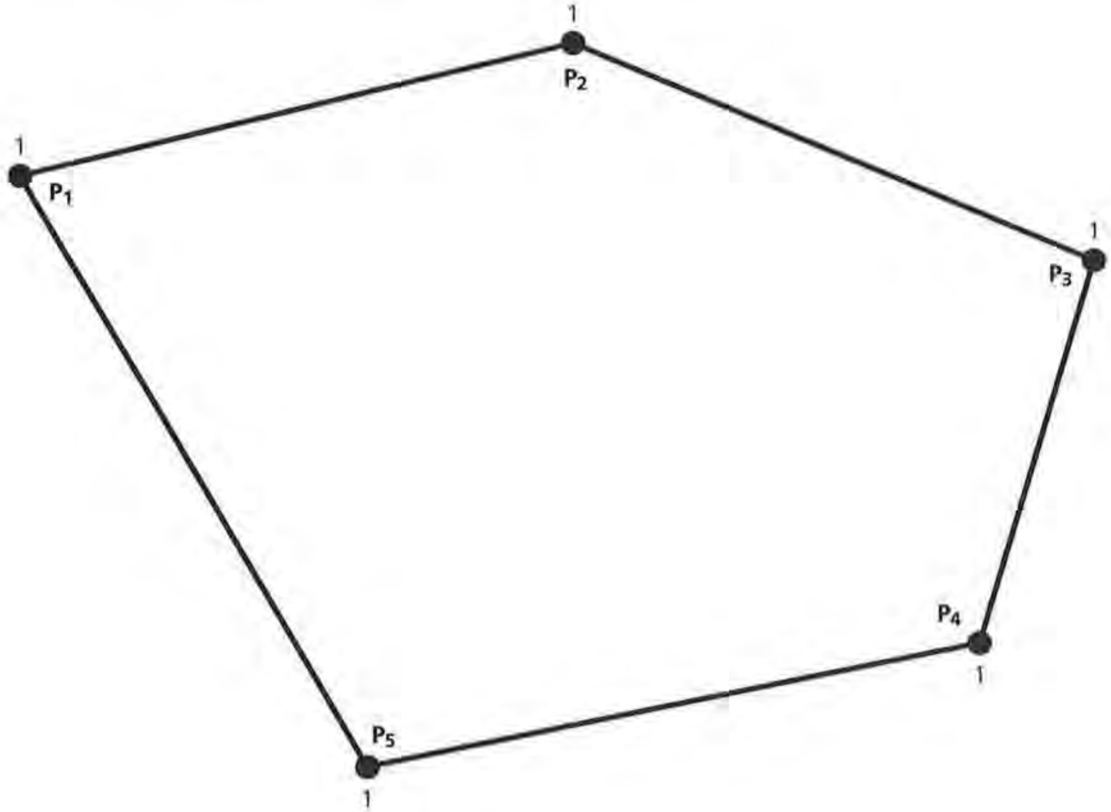
5. Why is this shape considered convex?

6. Are there any special characteristics you previously studied that could describe this pentagon? If yes, explain the characteristics you identified.

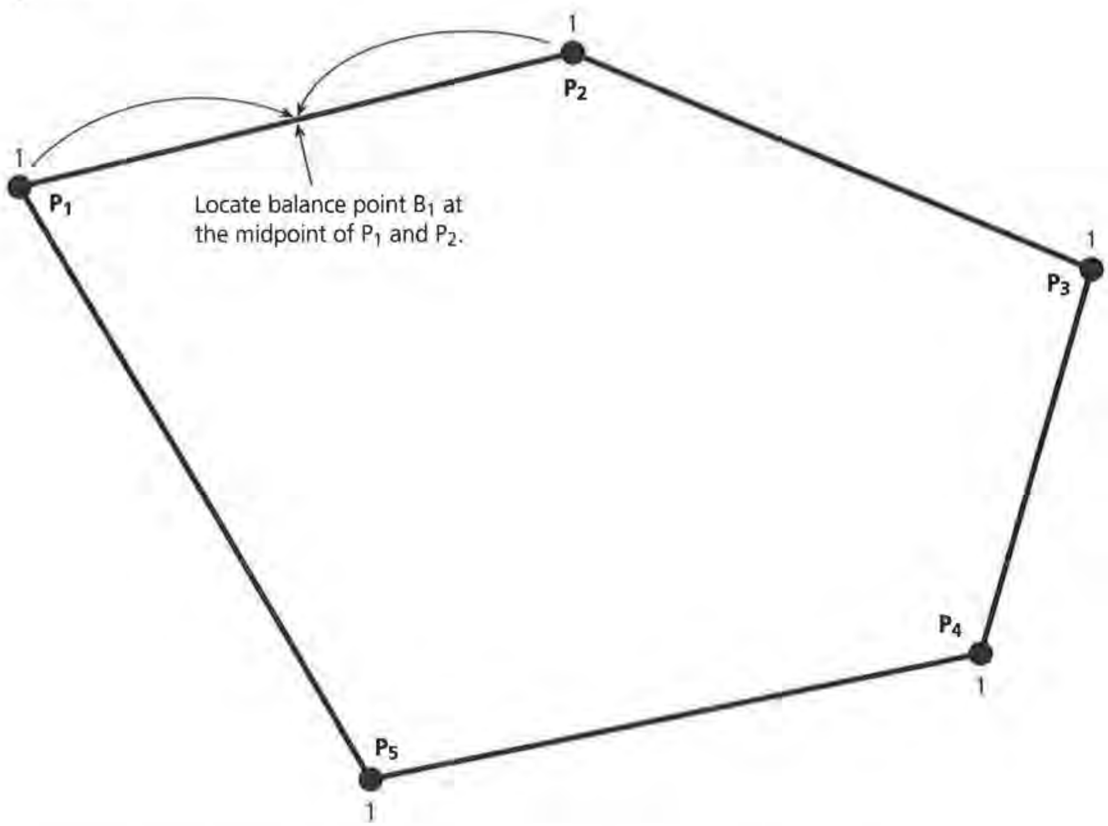
You will be investigating several methods to determine the center of balance. Remember the center of balance primarily refers to the center of the distribution of the objects, not the actual cut-out pentagon.

Generalizing a Balance Point

Imagine that five raisins of equal weight are located on the following "map" of the polygon. The raisins will be identified by the vertices of the polygon, or $\mathbf{P_1}$, $\mathbf{P_2}$, $\mathbf{P_3}$, $\mathbf{P_4}$, and $\mathbf{P_5}$. The value "1" on the diagram below indicates the weight of the point-mass (in this case, 1 indicates the weight of 1 raisin).



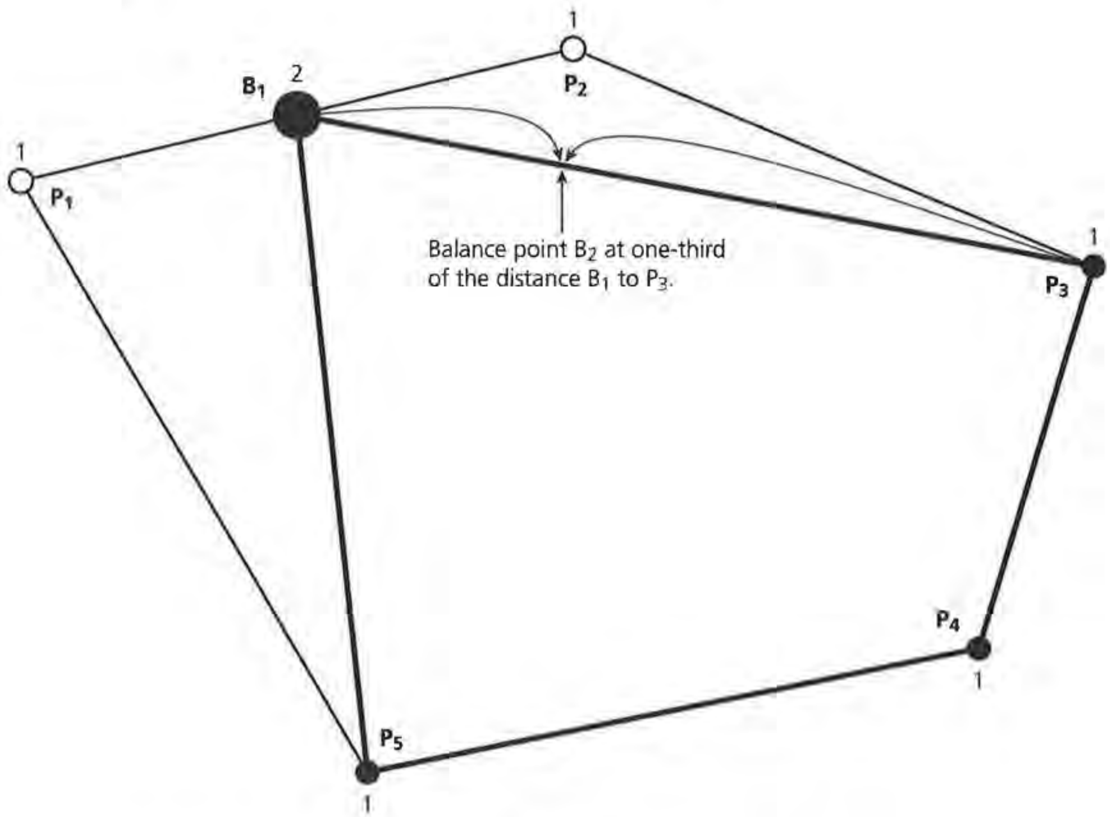
7. Use a ruler to measure the length of $\overline{P_1P_2}$. Previous investigations with two objects of equal weight indicated the balance point for two objects of equal weight is the midpoint of this segment. Determine the midpoint with your ruler and mark this point as B_1 ("balance point 1") on your cut-out model. This step is summarized in the following diagram.



If raisins were actually used in this example, B_1 would represent the location where the two raisins located at P_1 and P_2 can be placed and still keep the original model in balance. Consider removing the raisins from positions P_1 and P_2 and piling them together at B_1. The point B_1 provides a new configuration of the figure. Points P_1 and P_2 have been "collapsed" to point B_1. The pentagon has been "collapsed" into a quadrilateral. However, B_1 represents a weight of two units (or two raisins).

8. With your ruler or straightedge, measure $\overline{B_1P_3}$. Two raisins are located at one end of this segment and one raisin is located at the other end. A balance point involving weights of two units and one unit is needed. (This was previously studied when investigating the balance point of a triangle's medians in Lesson 5.) Let B_2 represent the location of this

balance point for the two raisins located at $\mathbf{B_1}$ and the one raisin located at $\mathbf{P_3}$. Determine the location of $\mathbf{B_2}$ with your ruler by marking a point one-third of the way from $\mathbf{B_1}$ to $\mathbf{P_3}$. This process is illustrated in the next diagram.



(If necessary, review the steps outlined in Lesson 5 to indicate the location of a balance point one-third of the distance from the object of weight 2.)

Consider removing the two raisins from $\mathbf{B_1}$ and the one raisin from $\mathbf{P_3}$ and piling the three raisins together at location $\mathbf{B_2}$. If $\mathbf{B_2}$ is an accurate location of the balance point for $\overline{\mathbf{B_1P_3}}$, the balance point of the original model of the raisins has not changed by collapsing the raisins. The quadrilateral has now collapsed into a triangle with the weight of three raisins at point $\mathbf{B_2}$.

Continue the process. Draw $\overline{\mathbf{B_2P_4}}$. The weight at $\mathbf{B_2}$ is three units and the weight at $\mathbf{P_4}$ is one unit. Similar to the work when the raisins were collapsed along the medians of a triangle, the balance point of this segment collapses three raisins at one endpoint and one raisin at the other endpoint.

Let $\mathbf{B_3}$ represent the point that balances the raisins along $\overline{\mathbf{B_2P_4}}$. Also, represent d_1 and d_2 such that:

$d_1 = B_2B_3$
$d_2 = P_4B_3$
and $d_1 + d_2 = B_2P_4$

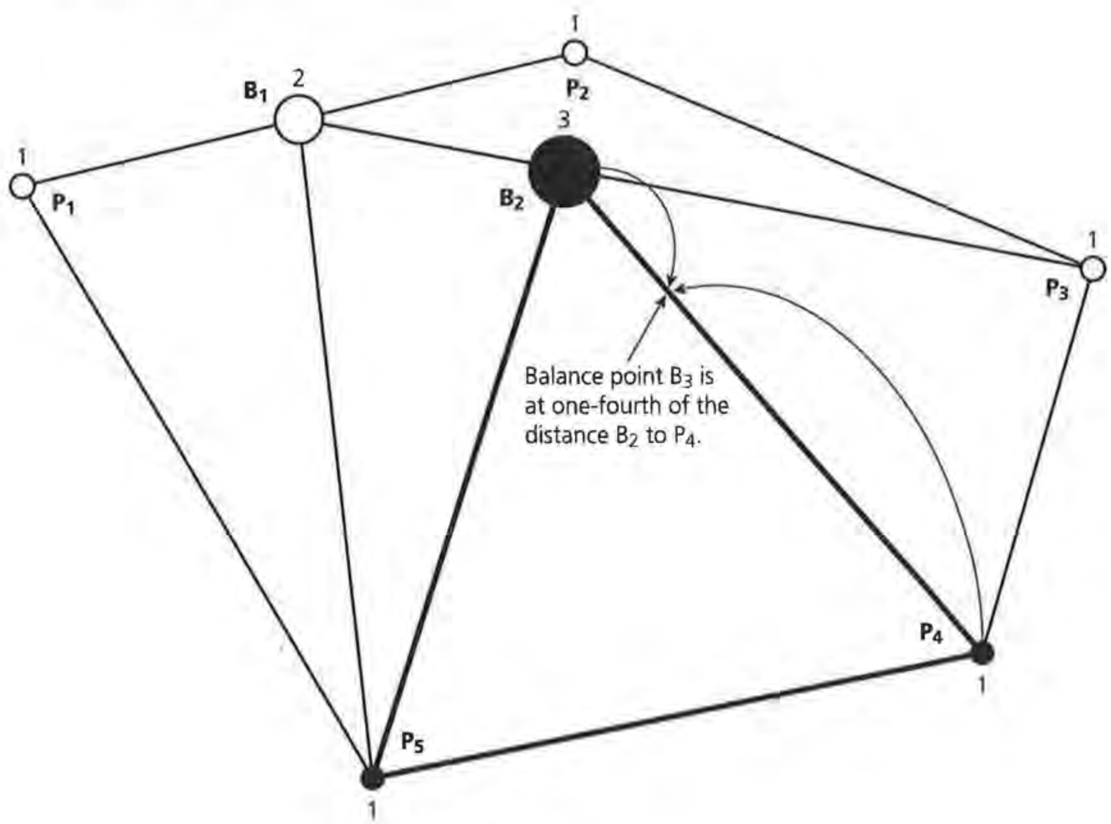
The balance at $\mathbf{B_3}$ indicates that:

$1d_2 = 3d_1$

Therefore,

$d_1 + d_2 = B_2P_4$
$\Rightarrow d_1 + 3d_1 = B_2P_4$ (By substitution)
$\Rightarrow 4d_1 = B_2P_4$
$\Rightarrow d_1 = \frac{1}{4} B_2P_4$

9. Measure segment $\overline{\mathbf{B_2P_4}}$. Three raisins are located at one end of the segment and one raisin is located at the other end. Let $\mathbf{B_3}$ represent the location of the balance point for the three raisins located at $\mathbf{B_2}$ and the one raisin located at $\mathbf{P_4}$. Determine $\mathbf{B_3}$ with your ruler by marking a point one-fourth of the way from $\mathbf{B_2}$ to $\mathbf{P_4}$. This process is illustrated in the diagram below.



10. The process of collapsing raisins is continued. Similar to the previous problems, define d_1 and d_2 to find B_4 (the balance point along $\overline{B_3P_5}$ with four raisins at B_3 and one raisin at P_5).

11. In addition to the balance described in 10, what does B_4 represent?

Balance Point Through Experimentation

12. Tape one raisin to each of the vertices of the original polygon cut out from the poster paper. Using the blunt end (or eraser end) of the pencil used in previous lessons, attempt to actually balance the model on the pencil (or fulcrum). This point represents an approximate balance point of the pentagon model. Label the point on the cut-out figure as point B (the "balance point").

13. Measure and record the distance from B to B_4. Conjecture the meaning of point B and the meaning of point B_4 regarding the original model of the distribution of raisins.

Balance Point Through Coordinate Geometry

14. Place the cut-out polygon on a coordinate grid provided with this lesson. Record the following data based on your placement of the polygon.

Point	*x*-value	*y*-value
P_1	______	______
P_2	______	______
P_3	______	______
P_4	______	______
P_5	______	______

15. Determine the centroid C $(\bar{x}, \bar{y})$ of this model by completing the calculations:

$$\bar{x} = \frac{x_1 + x_2 + x_3 + x_4 + x_5}{5} =$$

$$\bar{y} = \frac{y_1 + y_2 + y_3 + y_4 + y_5}{5} =$$

16. Locate the centroid on the coordinate grid. Label this point as C. Also label on the coordinate grid the points B and B_4.

SUMMARY

The balance point of objects of equal weight located on a plane can be estimated by designing a model and balancing the model on a fulcrum (or a pencil as used in this lesson). The balance point can also be estimated by placing the model on a coordinate grid and calculating the centroid. Finally, the balance point can be estimated by collapsing the objects (or the raisins) as demonstrated in this lesson.

Practice and Applications

17. The model used in this lesson was a convex polygon.

a. Would determining an estimate to the balance point by collapsing the raisins present a problem with a concave model? Explain.

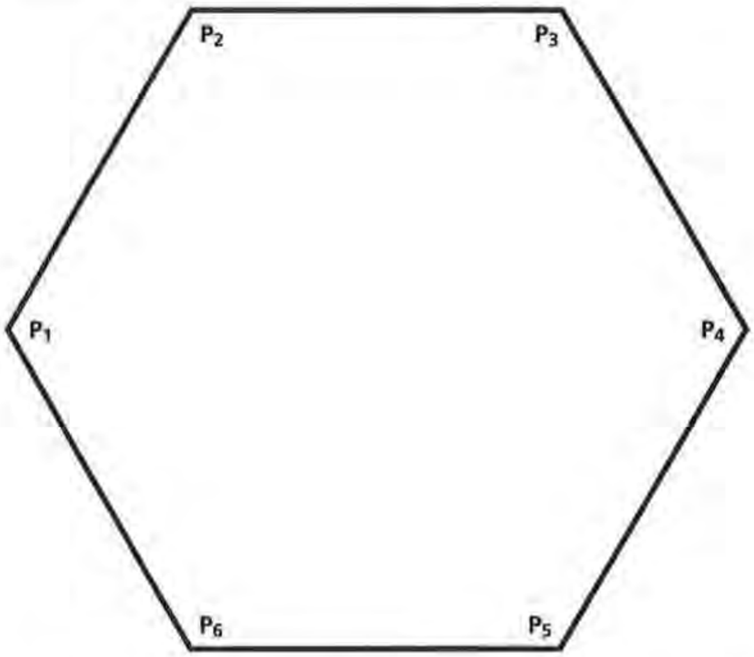
b. Which method of estimating a balance point might be complicated if the model presented were a concave polygon? Explain.

18. Return to the quadrilaterals studied in Lesson 6. If you have not developed a cutout on poster paper of quadrilateral 4 ("**Nothing Special**" model), develop one. Using your ruler, determine the position of the balance point by the process described as collapsing the raisins. Label this point as $\mathbf{B_3}$ on the cut-out form.

19. Examine $\mathbf{B_3}$ from question 18 and the balance point obtained in Lesson 6 (or the intersection of the midpoints of opposite sides of the quadrilateral). Label this point as **B** on the cut-out form if it was not previously labeled. Measure and record the distance between points **B** and $\mathbf{B_3}$.

20. Determine the balance point of **Quadrilateral 3** (the **Boomerang** model) by the method of collapsing the raisins. Compare the point estimated by this method with the balance point determined in Lesson 6.

Consider this polygon.



21. What is different about this polygon compared to the model studied in this lesson?

22. Consider a model in which a raisin is taped to each of the vertices of this new polygon. Without using any of the methods presented, sketch (or trace) this polygon and estimate the location of the balance point. (Label your estimate as point E.)

23. What did you consider in making your estimate of the balance point?

24. Determine a balance point for this model by the method outlined in this lesson as "collapsing the raisins." Provide a label for the point you determined by this method. How did this point and your estimate compare?

25. The word *center* is most commonly associated with a circle. How would the center of a circle apply in this example?

Weighted Means Revisited

Examine again the pentagon cut out of the poster paper in Lesson 7. How do you think the balance point of the pentagon model would change if the weight placed at vertex P_1 was doubled?

Tape a second raisin to the model at P_1 and balance it with the pencil. Did it change as you predicted?

A more precise location of this balance point is needed to determine the effect of the increased weight at P_1. What makes the calculation of a centroid more difficult if unequal weights are located at the vertices?

OBJECTIVES

Determine the center of balance by calculating the weighted means of the *x*- and *y*-coordinates of objects distributed on a plane.

Interpret the balance point by weights and locations.

Using a pencil to balance a pentagon with a raisin taped at each vertex was a relatively good model to estimate a center of balance of five weights. To what extent does changing the weight of the raisins or their locations on a plane alter the location of the balance point? What if the weight taped at one vertex of the pentagon model was doubled by taping a second raisin over the one already located there? How would this additional weight change the balance point? Similarly, what if a weight were completely removed from a vertex? How would this change the balance point? An earlier lesson taped several raisins at a specific location on a ruler. Increasing the weight at this point altered the location of the fulcrum to balance the new arrangement of raisins. Several questions were investigated related to that balance point. This lesson examines similar questions using a two-dimensional model.

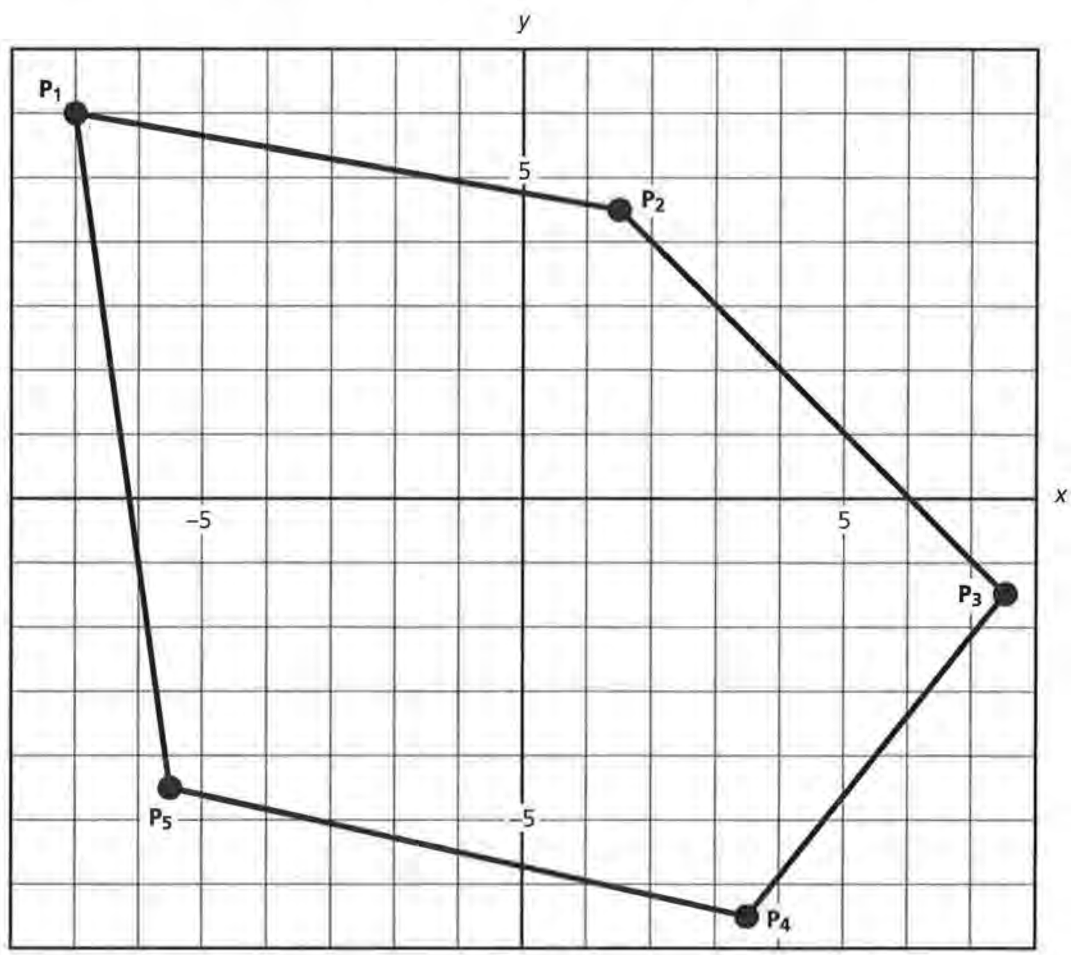
INVESTIGATE

Some Raisins Are Bigger than Others

In this lesson, each vertex will be weighted by *an unequal number of raisins.* Create a total pile of four raisins (one on top of each other) and tape this stack as close as possible to vertex P_1. Similarly, make a total stack of three raisins at P_2, one raisin at P_3, one raisin at P_4, and one raisin at P_5. Although the vertices still trace a pentagon, this new model will have a different center of balance than the model observed in Lesson 7. How would the new center of balance be determined?

Discussion and Practice

1. Estimate the new center of balance by balancing this new model with the pencil. Again, carefully move the pencil on the poster paper until this altered arrangement of raisins balances. How is this center of balance different from the center observed on the previous lesson? Describe the effect the additional raisins produce on the center of balance.



Consider the above placement of the pentagon model in an xy-coordinate system.

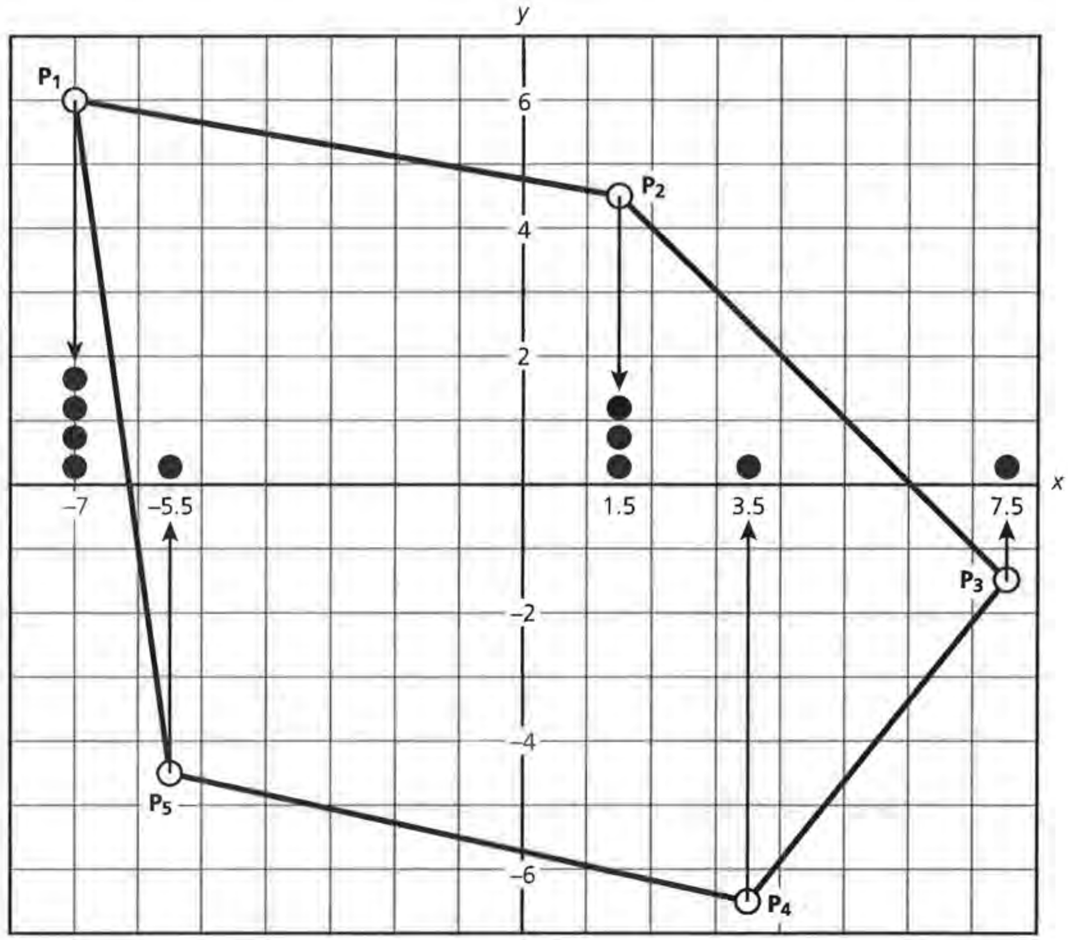
2. Your previous work with a center demonstrated how a balance point equally distributes the ***weighted distances*** along a ruler or number line. A similar balance is suggested by this model except in two dimensions.

   a. What horizontal or vertical distance is considered four times as great in this example than in the model presented in Lesson 7? Why?

   b. What vertical or horizontal distance is three times greater as a result of the weighted vertices? Why?

Location of the Center of Balance

In addition to balancing the pentagon model on the pencil, the location of the center of balance can be estimated by other methods. The calculation of the centroid as previously explained in this module will provide an estimate of this center. Examine the effect produced by the unequal weights taped to each vertex along the x-axis. Based on the placement of the figure as previously indicated, an illustration of the weighted values along this axis is illustrated in the following diagram:



To illustrate the effect of the weights taped at each vertex, the raisins described in this model are "dropped" to their horizontal locations along the x-axis. For example, the four raisins taped to P_1 are stacked on the x-axis at the x-coordinate of P_1. The x-axis now resembles the number line illustrated in Lesson 4. The weighted means of the x-values determines the horizontal position of the new center point. A similar illustration could be developed along the y-axis to represent the vertical position of the center point.

3. Copy and complete the following table based on this particular placement of the pentagon:

Points P_i (x_i, y_i)	Number of Raisins W_i $\boxed{L_1}$	x-coordinate values x_i $\boxed{L_2}$	y-coordinate values y_i $\boxed{L_3}$	Weighted x-values $W_i x_i$ $\boxed{L_4 = L_1 * L_2}$	Weighted y-values $W_i y_i$ $\boxed{L_5 = L_1 * L_3}$
P_1 (–7, 6)	4	–7	6	–28	24
P_2 (1.5, y_2)	3	1.5	y_2	4.5	$3y_2$
P_3 (7.5, y_3)	1	7.5	______	______	______
P_4 (3.5, y_4)	1	3.5	______	______	______
P_5 (–5.5, y_5)	1	–5.5	______	______	______

Five points are involved in this example. The weighted mean of the x-values would be represented by the following summation:

$$\bar{x} = \frac{\sum_{i=1}^{5} W_i x_i}{\sum_{i=1}^{5} W_i} = \frac{4(-7) + 3(1.5) + 1(7.5) + 1(3.5) + 1(-5.5)}{(4 + 3 + 1 + 1 + 1)}$$

The x-coordinate of –7 is multiplied by 4 as $\mathbf{P_1}$ is weighted down with four raisins. Similarly, the x-value of 1.5 for $\mathbf{P_2}$ is multiplied by 3 as three raisins are located at this location.

4. The weighted mean for this model is based on dividing the weighted distances by 10. What does the 10 represent?

5. Another way to represent this mean is:

$$\bar{x} = \frac{-7 + -7 + -7 + -7 + 1.5 + 1.5 + 1.5 + 7.5 + 3.5 + -5.5}{10}$$

Why is –7 added four times in this calculation of the mean $\bar{x}$?

6. Complete the calculation of $\bar{x}$.

The weighted mean of the y-coordinate values is used to determine $\bar{y}$ of the center of balance. A summation of the data recorded in the table to calculate $\bar{y}$ is summarized below:

$$\bar{y} = \frac{\sum_{i=1}^{5} W_i y_i}{\sum_{i=1}^{5} W_i} = \frac{4(6) + 3(y_2) + 1(y_3) + 1(y_4) + 1(y_5)}{(4 + 3 + 1 + 1 + 1)}$$

7. Complete this calculation of $\bar{y}$.

8. Combine your results from problems 6 and 7. Mark on the cut-out model of the pentagon constructed in Lesson 7 the centroid of this ***weighted mean*** example, or C_w $(\bar{x}, \bar{y})$.

9. Your pentagon model or cutout is rather "crowded" with points. In Lesson 7, estimates of the balance point with one raisin taped to each vertex were identified as B_4, C, and B. Describe how C_w compares to any of these estimates. Did you expect a change in the position of C_w when compared to B_4, C, or B? Explain your answer.

10. How does C_w compare to the location of the balance point obtained with the pencil in problem 1? Was the balance point in problem 1 a good estimate of the new centroid? Explain your answer.

Using a Spreadsheet or Calculator

Applications involving weighted means will be more extensively developed in the population models presented in Lessons 9 and 10. Organizing the data involved in these calculations is important in order to determine an accurate centroid. The pentagon example provides an excellent problem to experiment with using either a general spreadsheet (or similar application program) or a calculator. The table presented in problem 3 had columns subtitled L_1, L_2, L_3, L_4, and L_5. Several calculators are equipped with LIST capabilities. The TI-83 (and models developed since the introduction of this calculator) identify the available data lists as L_1 to L_6. Complete the steps as outlined. Although the steps specifically refer to the TI-83 calculator, study the directions as presented. Modifications for other calculators will require understanding the layout of that specific calculator.

11. Enter the number of raisins taped to each vertex in your first list, (or L_1 in the TI-83 setup), the x-values of the vertices in the second list, or L_2, and the y-values of the vertices in the third list, or L_3.

For the TI-83, this is accomplished in the following way: Hit the $\boxed{\text{STAT}}$ key. If lists have been previously entered into the calculator, you may need to select the ClrList option of this menu and identify the lists to be cleared. Otherwise, select the EDIT menu option. This allows you to enter your data. A general summary of these steps is developed below. Again, different calculators will require a modified process in developing lists of the data.

If you need to clear the data from the lists, then . . .	If it is not necessary to clear the lists, then . . .
[STAT] → ClrList [ENTER] ClrList $L_1, L_2, L_3, L_4, L_5, L_6$ [ENTER] [STAT] → EDIT enter data as indicated in the appropriate list columns.	[STAT] → EDIT enter data as indicated in the appropriate list columns.

(Note: to back up, start over, or re-enter the data, you might need to return to the beginning of this process. To start again, hit [2nd] [MODE]. The [2nd] key indicates the command written on top of the raised [MODE] key will be executed or entered. In this case, [2nd] [MODE] indicates the [QUIT] instruction will be executed. This instruction clears the screen and allows you to return to the [STAT] options, or, to re-enter the EDIT option of LISTS.

12. If the data is correctly entered in lists L_1, L_2, and L_3, you are now ready to "program" the weighted values into your fourth and fifth lists, L_4 and L_5. This can be done in different ways. One of the options available for the TI-83 involves the following:

With the lists visible in your window, use the arrow keys, [►] and [▲], to move the cursor to the top of L_4. If successful at this point in the process, L_4 will be highlighted. The bottom of the screen should display:

$L_4 =$

Using [2nd] 1, and so forth, enter the following formula for L_4:

$L_1 * L_2$ [ENTER]

This formula multiplies the number of raisins and the x-coordinate values for each point. Specific values should now be displayed in the L_4 list.

13. In a similar way, calculate the values for L_5. Use the arrow keys to position the cursor on top of L_5. Enter the following formula for L_5:

L_1*L_3 $\boxed{\text{ENTER}}$

The values representing the number of raisins multiplied by the y-coordinate values for each point should now be displayed in the L_5 list.

14. To find the coordinate values of the centroid, or C $(\bar{x}, \bar{y})$, the following summations are required:

$$\bar{x} = \frac{\text{sum of } L_4}{\text{sum of } L_1} \text{ and } \bar{y} = \frac{\text{sum of } L_5}{\text{sum of } L_1}$$

The following steps should be followed to calculate the above values on a TI-83.

- Clear the screen: 2nd $\boxed{\text{MODE}}$ or [QUIT].
- Hit 2nd $\boxed{\text{STAT}}$ or [LIST].
- Select MATH from the menu options.
- Select **5:sum(** from the MATH menu options and then enter L_4).

On the home screen you should see sum (L_4). This expression will determine the sum of the values entered in L_4.

- Now hit the divide key and you should see:

  sum (L_4) /

- Hit 2nd $\boxed{\text{STAT}}$ or [LIST].
- Select MATH from the menu options.
- Select **5:sum(** from the MATH menu options and then enter L_1).

You should now see **sum (L4) / sum (L1)**.

This indicates the sum of the values entered in L_4 will be divided by the sum of the values entered in L_1.

The resulting value is $\bar{x}$.

15. Develop similar steps to determine the value of $\bar{y}$ using a calculator or a computer program.

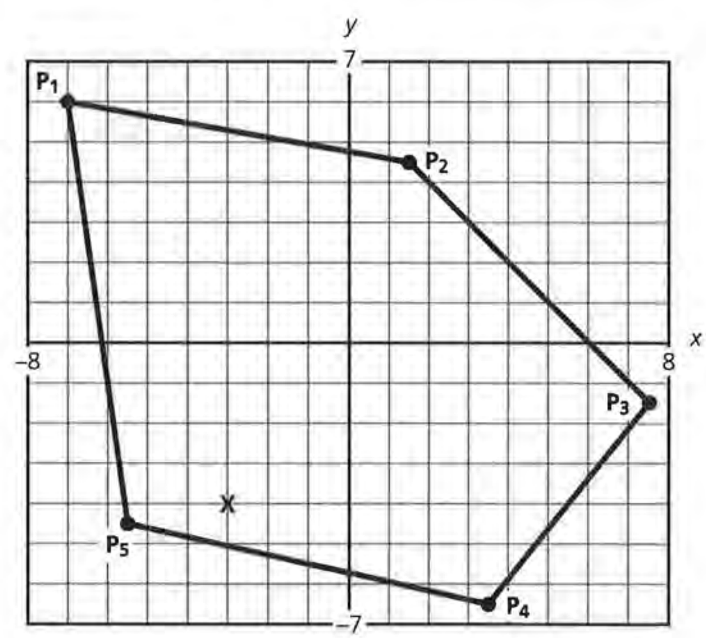
(Note: if problems develop in data entry or sequencing the above steps, hit 2nd $\boxed{\text{MODE}}$ or [QUIT] to clear the screen and re-enter the steps explained in the previous problem.)

SUMMARY

A balance point for a distribution of weighted objects on a plane is determined by the weighted means of the coordinate values. The difficulty of designing a model to pinpoint the point-mass objects makes the center of balance by experimentation less accurate here than in the previous models demonstrated. The location of the centroid is primarily influenced by the location of the "heavier" points.

Practice and Applications

16. Consider the pentagon investigated in this lesson. *Hypothesize* the number of raisins located at each vertex of this shape that would place the center of balance at approximately the location identified as X for each of the following examples:

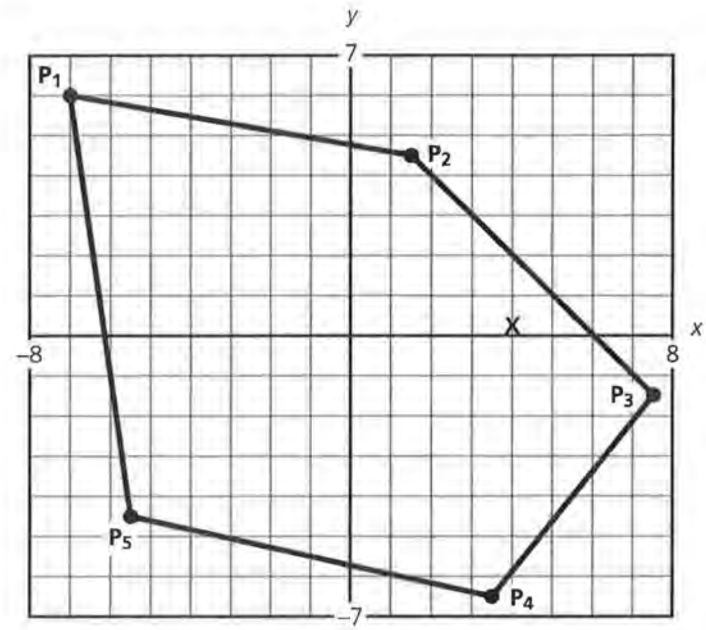


a. Estimate the number of raisins at each vertex for this example:

P_1 P_2

P_3 P_4

P_5



b. Estimate the number of raisins at each vertex for this example:

P_1 P_2

P_3 P_4

P_5

17. Test the accuracy of each of your estimates for problem 16 by calculating the centroid; organize the data for your estimate in a table as outlined below. Copy and complete the following table for each example:

Points P_i (x_i, y_i)	Number of Raisins W_i $\boxed{L_1}$	x-coordinate values x_i $\boxed{L_2}$	y-coordinate values y_i $\boxed{L_3}$	Weighted x-values W_ix_i $\boxed{L_4 = L_1 * L_2}$	Weighted y-values W_iy_i $\boxed{L_5 = L_1 * L_3}$
P_1 (–7,6)	______	______	______	______	______
P_2 (1.5, ?)	______	______	______	______	______
P_3 (7.5, ?)	______	______	______	______	______
P_4 (3.5, ?)	______	______	______	______	______
P_5 (–5.5, ?)	______	______	______	______	______

18. Describe how you could improve your estimates for each example.

Population Centers

Estimating a balance point for objects located on a plane is closely related to finding a population center. This section of *Exploring Centers* will focus on the calculation and significance of a population center.

As you read and study this unit, attempt to design the map of a "small country." Locate on your map at least five major cities, several highways, at least one major airport, a capital city, and at least five small towns. The capital city could be one of the five major cities or one of the five small towns. Locating rivers, lakes, mountains, and other natural resources is optional. Be prepared to explain how you selected the location of your airport and your capital city. Also, be prepared to explain why you routed the highways as indicated on your map.

Give your country a name, a flag, and a total population. In addition, describe the population by summarizing the average age, major employment categories, general educational information, recreational activities, and any other characteristics you think are important. As a guide, study the data presented on countries presented in a world almanac.

LESSON 9

Finding a Population Center

Suppose there are people, instead of raisins, located at the vertices of a very large pentagon. Would your method change for finding the center of balance?

Why might this center of balance, or population center, be important?

What type of questions does the location of a population center answer?

The balance point models developed in the previous lessons assumed the weight of each raisin (or object) was equal. If several raisins were located at each vertex, then a weighted mean was calculated. This equalizes the effect of each raisin on the balance of the objects.

A *population center* has a similar interpretation. Instead of balancing the weight of objects related to distance, however, a population center balances the "number of people" related to distance. A population center gives "equal status" to each person based on his or her location on a plane. It represents the location where the number of people and their respective distances are balanced. This lesson and Lesson 10 attempt to develop and explain the significance of a population center.

OBJECTIVES

Interpret a population center as a measure of balance.

Set up and calculate a population center.

Interpret a population center as an average of location and population.

Describe the appropriateness of applying a population center to a real-world problem.

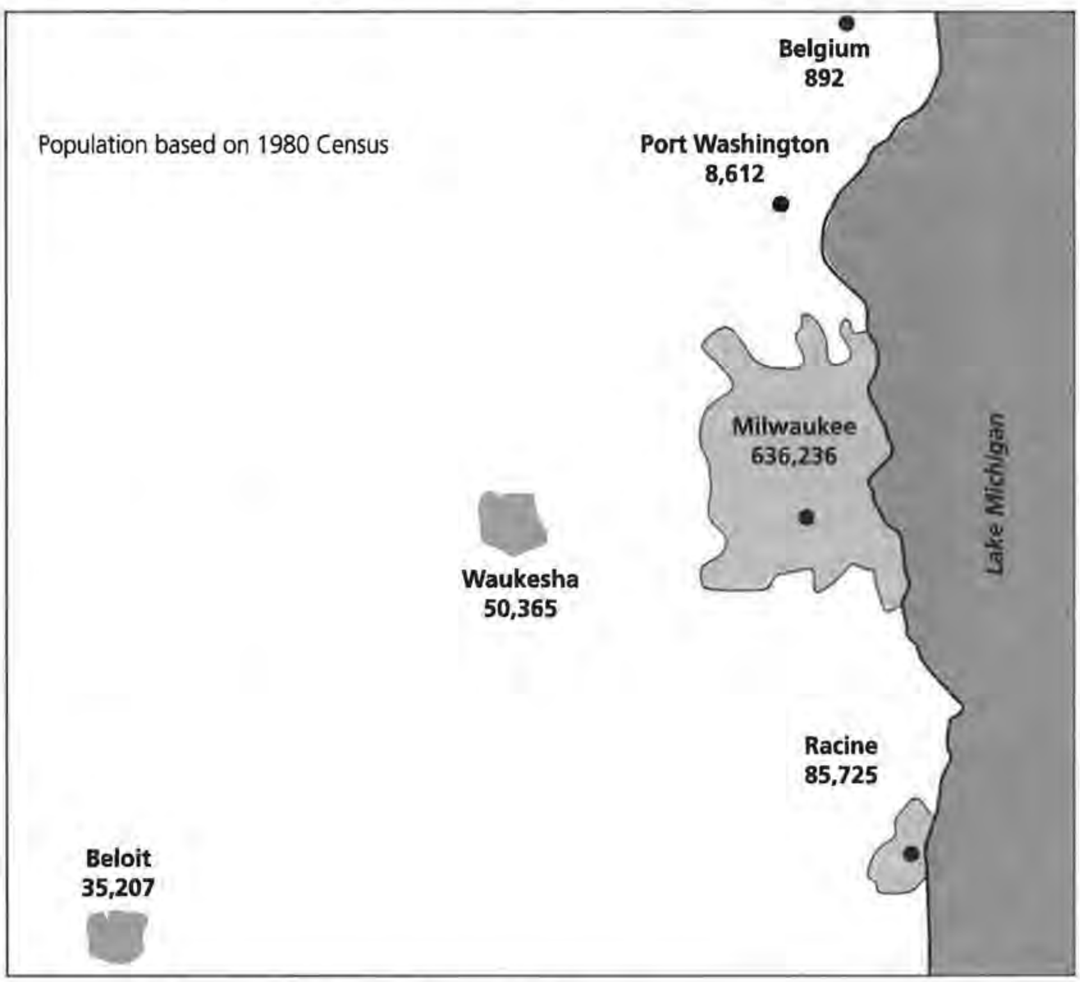
INVESTIGATE

Population Centers

A population center is the "balance point" of a distribution of people. Population statistics are extensively studied and analyzed at the local, national, and global level. The United States Constitution directs that an actual count of the citizens of this country must be made every 10 years. The Bureau of the Census is the designated agency to carry out a U.S. census at the start of each decade. Each completed census provides volumes of data to interested citizens and political groups.

Discussion and Practice

Pretend you are an important political person back in the 1980s. You are appointed by the Governor of Wisconsin to head a committee to determine the location of an important job service agency in that state. You and your committee are responsible for helping the people residing in the communities of Milwaukee, Waukesha, Port Washington, Belgium, Beloit, and Racine. A map of this area of Wisconsin is provided below. Also included in this sketch are the 1980 population statistics that the committee will use in making a decision. The recommendations of your committee are expected to service the people of this area for at least the next 20 years. A review of your decision will be made in 1990 and again in 2000. You have an



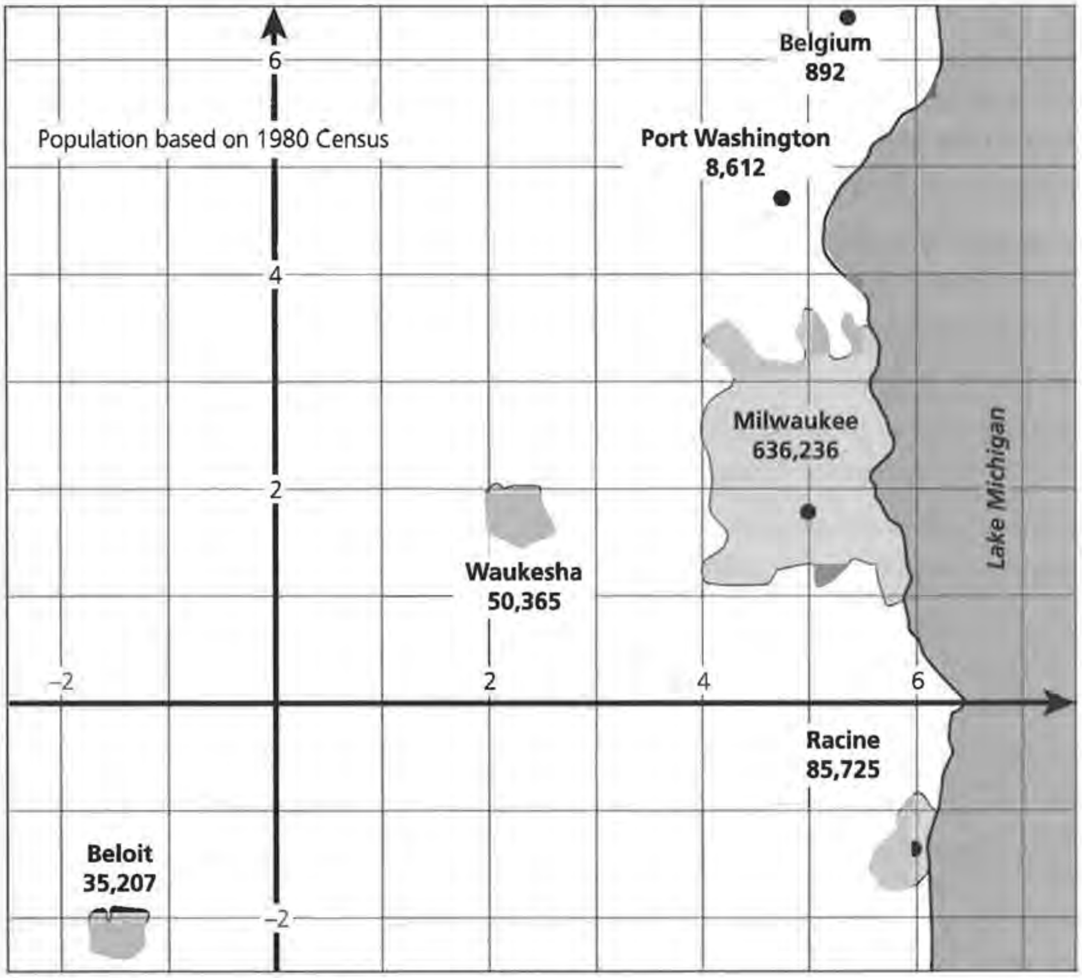
Source: 1980 U.S. Department of Commerce, Bureau of the Census interest in running for governor at some future date and feel this appointment is an opportunity to demonstrate your leadership skills.

1. Describe at least three factors you and your committee should consider in making a determination concerning the location of the job service agency.
2. Representatives of the six communities identified as the primary recipients of this service might recommend that the agency be located in their respective communities.
   a. Explain why a decision to locate the agency in Beloit might be "unfair."
   b. Explain why a decision to locate the agency in Belgium might be considered "unfair."
   c. Explain why a decision to locate the agency in Racine might be considered "unfair."
3. If the advisory group attempts to base their decision on fairness to the people located in these communities, what factors do you think are most critical to consider?

Locating a Population Center

The general location of each community is identified on the following map by a point. The collection of the communities (or points) resembles the model of a polygon. Replacing the number of raisins at each vertex with the number of people counted at each community allows you to begin thinking about a balance point. As the population of each community is not equal, calculating a balance point or population center would be similar to the model developed in Lesson 8 for weighted means.

Consider calculating a population center by placing a coordinate grid over the map below:



Source: 1980 U.S. Department of Commerce, Bureau of the Census

4. Based on this particular coordinate grid, determine approximate x and y values for each of the communities.

Community	*x*-coordinate value	*y*-coordinate value
Belgium	______	______
Port Wash.	______	______
Waukesha	______	______
Milwaukee	______	______
Beloit	______	______
Racine	______	______

5. A population center organizes the population and coordinate values of the communities in the same way the weights and coordinate values of the raisins were organized in Lesson 8. Copy and complete the following chart to determine the population center. (Note: use of a calculator or spreadsheet is encouraged but not required.)

Community	Population P_i	x-coordinate value (x_i)	y-coordinate value (y_i)	$P_i x_i$	$P_i y_i$
Belgium	892	______	______	______	______
Port Wash.	8,612	______	______	______	______
Waukesha	50,365	______	______	______	______
Milwaukee	636,236	______	______	______	______
Beloit	35,207	______	______	______	______
Racine	85,725	______	______	______	______

6. The population center is described by the following weighted means:

$$\bar{x} = \frac{\sum_{i=1}^{6} P_i x_i}{\sum_{i=1}^{6} P_i} \text{ and } \bar{y} = \frac{\sum_{i=1}^{6} P_i y_i}{\sum_{i=1}^{6} P_i}$$

a. What does $\sum_{i=1}^{6} P_i$ represent? (Compare this to $\sum_{i=1}^{5} W_i$ in the pentagon model developed in the previous lesson.)

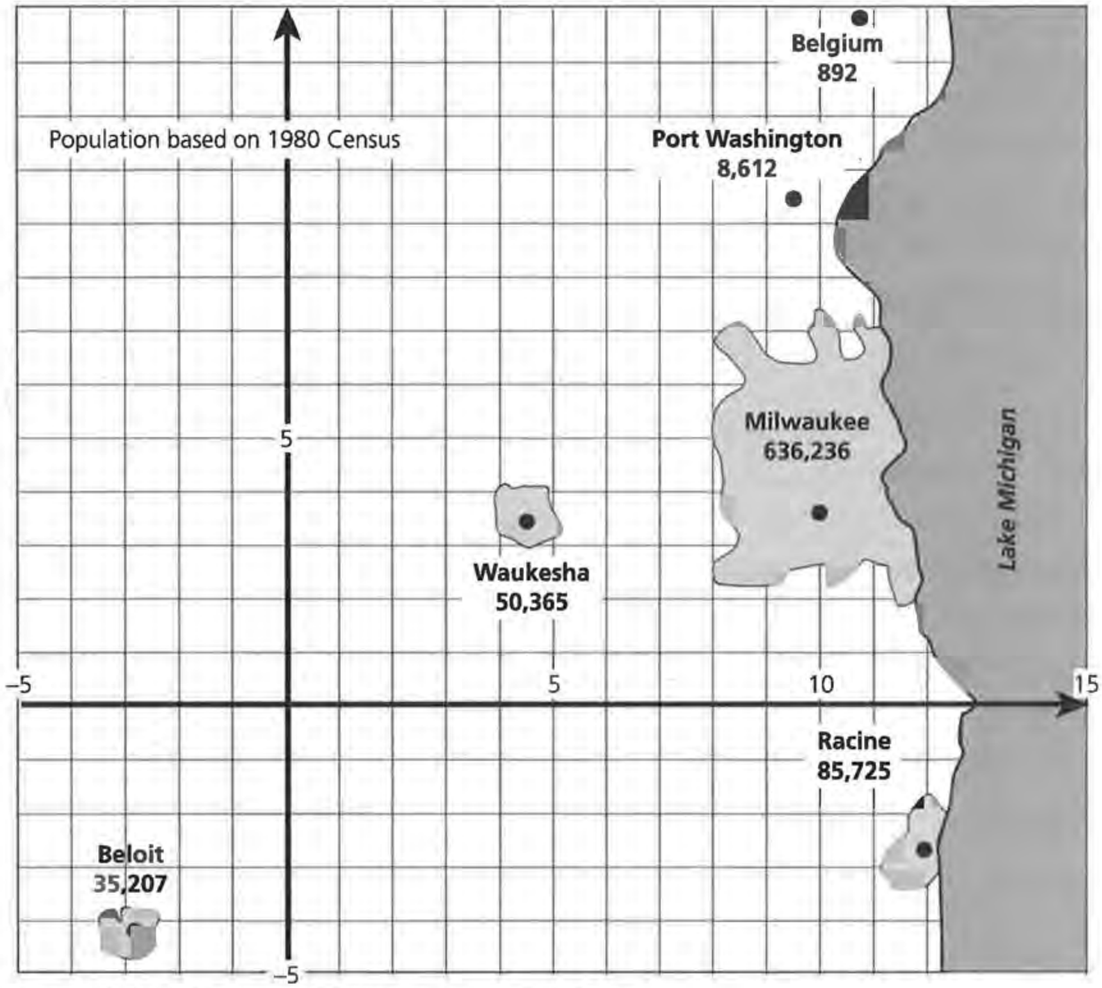
b. What does $\sum_{i=1}^{6} P_i x_i$ represent? (Compare this to $\sum_{i=1}^{5} W_i x_i$ in the pentagon model.)

7. Complete the calculations to determine the population center, or **P** $(\bar{x}, \bar{y})$, for the map provided.

8. Does the location suggested by problem 7 seem to be a "fair" location for the job service agency? Why or why not?

Will the design or placement of a specific coordinate grid change the location of the population center?

9. A more detailed coordinate system was designed to determine the population center. A revised map of the communities is provided below.



Source: 1980 U.S. Department of Commerce, Bureau of the Census

Recalculate a population center by collecting and calculating the following information from this revised coordinate grid. Copy and complete this table:

Community	Population P_i	x-coordinate value (x_i)	y-coordinate value (y_i)	$P_i x_i$	$P_i y_i$
Belgium	892	______	______	______	______
Port Wash.	8,612	______	______	______	______
Waukesha	50,365	______	______	______	______
Milwaukee	636,236	______	______	______	______
Beloit	35,207	______	______	______	______
Racine	85,725	______	______	______	______

10. Determine the population center Q ($\bar{x}$, $\bar{y}$), from this table.

11. Compare the locations of the two population centers derived from each example, P and Q. Would you expect the centers to be similar? Why or why not?

12. If the population center were to be suggested as the location of the agency, do you think each community would consider this recommendation "fair"? In what way does the population center approach seem to address some of the concerns of fairness?

The "Belgium Problem"

An incomplete table of the populations of the communities involved in this decision was developed. Consider the following incomplete table:

Community	Population P_i	*x*-coordinate value (x_i)	*y*-coordinate value (y_i)	P_ix_i	P_iy_i
Port Wash.	8,612	_____	_____	_____	_____
Waukesha	50,365	_____	_____	_____	_____
Milwaukee	636,236	_____	_____	_____	_____
Beloit	35,207	_____	_____	_____	_____
Racine	85,725	_____	_____	_____	_____

13. What data were omitted from this table? Is this fair?

14. Determine a population center for the five communities listed in this incomplete table. Use either the first or the second grid to complete the *x*- and *y*-coordinate values for each community.

15. Compare the population center derived from this incomplete table with the center derived when all six communities were considered. How does the incomplete data table change the recommendation of the advisory group?

16. Describe what you think is meant by the title of this investigation ***"The Belgium Problem."***

17. Consider the omission of the city of Milwaukee data from the table listing the six communities. Would this omission change the location of the center of the population? Estimate where the center of population would be located if the city of Milwaukee were not considered in the population distribution.

SUMMARY

The population center represents a location balancing the number of people and their respective locations. Each person is considered "equal" in the calculation of this center. Similar to the previous study of a balance point, the location of the population center is influenced by a region's specific populations and their respective locations.

Practice and Applications

18. Evaluating your 1980 decision is important! The 1990 Census of the six communities studied in this lesson is provided in the following table. Copy the table below. Use one of the two coordinate grids to help you complete this table.

Community	Population P_i	x-coordinate value (x_i)	y-coordinate value (y_i)	$P_i x_i$	$P_i y_i$
Belgium	1,405	______	______	______	______
Port Wash.	9,338	______	______	______	______
Waukesha	56,958	______	______	______	______
Milwaukee	628,088	______	______	______	______
Beloit	35,573	______	______	______	______
Racine	84,298	______	______	______	______

19. Before calculating the revised population center using the 1990 population figures, estimate the location of the 1990 population center as a coordinate point on the grid.

Explain what factors you considered in making your estimate.

20. Based on the 1990 population figures, determine the population center **P** $(\overline{x}, \overline{y})$.

21. Describe how the population center changed from 1980 to 1990.

If the location of the job service agency were based on the 1980 populations, would the six communities be similarly served by the agency in 1990? Explain your answer. Was your 1980 decision a good one?

22. Consider the following investigations. For each item, explain whether or not a population center would be considered. In addition, indicate what other factors might be involved in each decision.

a. Location of a church or temple or synagogue for a large congregation.

b. Location of a school for a growing school district.

c. Location of a shopping mall for a suburban community.

d. Location of an airport to service three large cities.

e. Location of a large recreational facility (for example, a stadium for a major league baseball or football team).

f. Location of public housing units for the elderly.

23. Using a map of your state or geographic area, determine the population center for the three or four largest communities of interest to you or your class.

24. Locate Gurnee, Illinois, on a map of Illinois and Wisconsin. This small community is the location of a very large recreational park, Six-Flags Great America®. Why would a very small community be selected as the location for one of the largest amusement parks in the United States?

25. Consider the previous map of the six communities involved in the investigations of this lesson. Someone hypothesized that it was possible for the center of population of the communities to be located in Lake Michigan! Could there be a population distribution of these six communities that would place the center of population in Lake Michigan? Explain your answer.

Before you complete this module, it is very important to more thoroughly understand a population center. For example, a population center is *not* the location that minimizes distances to a specific point for each of the people in the model. Although frequently close to this point, finding a location that minimizes distances represents another type of center. Stay tuned. Investigations in Lessons 11 to 13 will help you understand other types of centers.

LESSON 10

Finding the Population Center of the United States

Do you think you could compute the population center of the entire United States?

Would you expect the population center of the U.S. in the year 1775 to be the same as the population center of the U.S. in the year 1850? Why or why not?

OBJECTIVES

Calculate a population center based on coordinate points and census populations.

Hypothesize population changes and the effects on the population center.

The population center for the six communities in southeastern Wisconsin was a manageable extension of the polygon models. Expand on this idea even more by imagining that you are in outer space looking down at the entire United States. Instead of viewing a small section of one state, you are now looking at an entire country! Where do you think the population center for the United States is currently located?

INVESTIGATE

Westward, Ho!

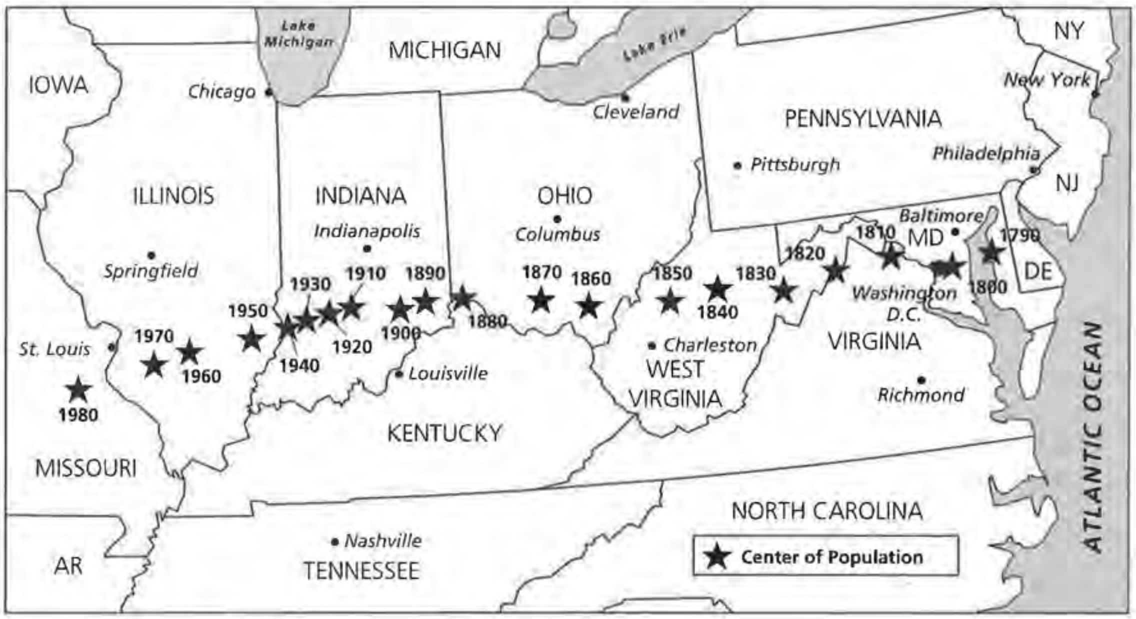
The U.S. Bureau of the Census determines the population center of the United States from the census data collected at the beginning of each decade. The map on the next page plots the population centers for each decade from 1790 to 1980, as determined by the Bureau of the Census. Consider how the population center of the United States has shifted over the years.

Discussion and Practice

1. The population centers have consistently moved westward. Explain this westward shift in the population center.

2. In addition to a westward shift, the population centers have also indicated a trend to the south, especially since 1910. What are your ideas on the gradual shift to the south and west?

3. Your goal in this lesson is to approximate the center of population for 1990. The location of this center as determined by the 1990 Census will be shared with you at the conclusion of this lesson. Based on the centers presented in the above map, where do you think the 1990 population center was located? Why did you select this location?



Source: *The Statistical Abstract of the United States,* 1991

4. Some decades indicate a greater shift of the country's population center than other decades. Investigate what historical events highlight the following decades that might explain the greater shifts in the population noted on the map:

a. 1870–1880

b. 1900–1910

c. 1970–1980

d. 1950–1960

5. As previously explained in Lesson 9, the population center is a measure that "equalizes" people based on their relative locations on the map of the United States. The population counts that were used in determining the population center were based on the results from the Bureau of the Census for each decade. Based on your understanding of the census, consider the following questions.

a. Do you think the Bureau of the Census is currently accurate in counting almost everyone in the country? Why or why not?

b. Do you think the Bureau of the Census was accurate in counting most of the people in the early decades of this map? Why or why not?

c. Describe people the Bureau of the Census might not have counted in the decades of 1790 to 1850.

6. Washington DC was a "planned city." The location of Washington DC was based on an honest attempt to identify a fair location for all the people in the country. Do you think the location of Washington DC was based on the population center of the country? Why or why not? (Research this question!)

How Is the Population Center Determined?

The population center of a country is found by calculating a point that "balances" the country's population. The magnitude of the numbers of people and the distances involved prevent you from developing an exact location of this center, however, the process presented in Lesson 9 could be used to determine an estimate of the population center. The process involved in estimating the 1990 population center will require a map of the 48 states, a list of the 1990 populations for each state, and an atlas or similar resource to locate state capitals.

Before a process is outlined to develop this project, a few questions need to be discussed specifically concerning Alaska and Hawaii.

7. Consider the populations of Alaska and Hawaii.

a. Why are Alaska and Hawaii interesting variations to the rest of the country?

b. Are there any other locations in the United States that might similarly complicate a representation of the population center for all of the citizens?

c. If yes, identify the locations and why they pose a problem.

8. Lesson 9 discussed the "Belgium problem." Essentially this problem indicated that the magnitude of the population of the larger communities was so much greater than Belgium's population that Belgium had little impact on the population center. The population sheet listing the 1990 population for each state indicates Alaska and Hawaii are small population states. However, they represent a different concern than simply a small population. In what way do Alaska and Hawaii require a different consideration than other states with small populations?

9. In order for Alaska and Hawaii to be included in the activities outlined in this lesson, an accurate map including Alaska and Hawaii is needed. This type of map is very difficult to develop.

   a. Examine a globe and locate Alaska. If you were given the population center of the 48 connected states, how would the addition of Alaska affect that center?

   b. Examine your globe again and locate Hawaii. If you were given the population center of the 48 connected states, how would the addition of Hawaii affect that center?

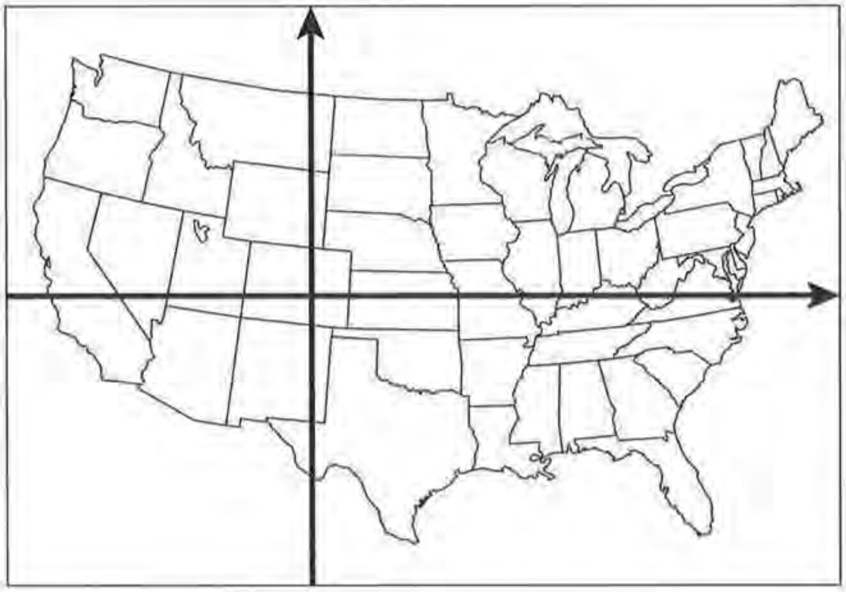
   c. The activities outlined in this lesson will conclude with an estimate of the population center for the 48 connected states. How would you suggest that center be adjusted to account for inclusion of Alaska and Hawaii?

The model in Lesson 9 formed a collection of points corresponding to the locations of the communities on the map. In addition, each point was weighted according to the population of the community. What will the points represent in the U.S. model? What locations will be used? What population statistics will be used?

The Big Picture

There are several methods that could be used to estimate the center for the 48 connected states. The first activity outlined is called "The Big Picture" method. It involves a lot of work and detailed investigations! This method is most manageable if developed with a spreadsheet. The following steps outline this method:

Step 1: Study a map of the United States and construct a coordinate system on the map. You may use *Activity Sheet 8* to help you. The specific placement of the origin and the x- and y-axes of this system is not important.



Step 2: Design *accurate* units on both your x- and y-axes. These units will be used to estimate the coordinates of the population centers for each state.

Step 3: Estimate a population center for each state by identifying the location of the state's capital. Many states modeled the federal government in planning a city as their capital based on an estimate of the population center of the state at the time statehood was granted. Clearly some state capitals are not good estimates of a population center, however, the effect on the coordinate values will not significantly change the estimation of the population center.

Step 4: Use the 1990 State Population Chart on *Activity Sheet 10* to help you complete the calculations for the weighted values of each state.

Although not all locations or people are represented by this process, the number of points located on a map indicate a major representation of the country's population. Complete the data requested in the chart on *Activity Sheet 10,* by using the coordinate values obtained from your map. (A spreadsheet is ideal for this method, although some graphing calculators such as the TI-83 are able to handle this large data collection.)

10. Using the data from your 1990 State Population Chart, estimate a population center by calculating the centroid of the set of points. The centroid C_1 $(\bar{x}, \bar{y})$ for the 48 states discussed would be based on the following summations:

$$\bar{x} = \frac{\sum_{i=1}^{48} P_i x_i}{\sum_{i=1}^{48} P_i} \text{ and } \bar{y} = \frac{\sum_{i=1}^{48} P_i x_i}{\sum_{i=1}^{48} P_i}$$

11. Locate the centroid on the United States map and label it as C_1. Adjust this point to include your estimate of including Alaska and Hawaii as discussed in problem 9. (Remember, Hawaii and Alaska are particularly important small states in the location of the country's population center.)

12. The location of the 1990 population center calculated by the Bureau of the Census is published in a number of sources, including special publications from the U.S. Department of Commerce, Bureau of the Census. The *Statistical Abstract of the United States* is an excellent source for comparing the 1990 center with previous centers. The population center as determined by the Bureau of the Census will be shared with you by your teacher. Compare your estimate with the center derived from the Bureau of the Census.

The Smaller Picture

Another method to consider will be discussed as the "smaller picture" of the population data. The "Belgium problem" indicated how the magnitude of the larger communities determined the location of the population center. Consider the following process in estimating a population center for the 48 connected states:

Step 1: Again construct an *xy*-coordinate system on the map of the United States.

Step 2: Estimate a population center for each of the 10 states with the greatest population. Here again, place a dot at the location of each state's capital.

13. Based on the ten population centers placed on the map, copy and complete the data sheet that follows.

Data for Determining the U.S. Center of Population "The Smaller Picture"

State (abbreviation)	Population P_i (in thousands)	*x*-coordinate value	*y*-coordinate value	$P_i x_i$	$P_i y_i$
______	______	______	______	______	______
______	______	______	______	______	______
______	______	______	______	______	______
______	______	______	______	______	______
______	______	______	______	______	______
______	______	______	______	______	______
______	______	______	______	______	______
______	______	______	______	______	______
______	______	______	______	______	______
______	______	______	______	______	______

14. Determine a centroid C_2 based on the ten points listed in the table.

$$\bar{x} = \frac{\sum_{i=1}^{10} P_i x_i}{\sum_{i=1}^{10} P_i} \text{ and } \bar{y} = \frac{\sum_{i=1}^{10} P_i y_i}{\sum_{i=1}^{10} P_i}$$

15. Locate this centroid on the United States map. Again, consider adjusting this point to include Alaska and Hawaii as summarized in problem 9. How does C_2 compare to C_1?

Practice and Applications

16. The census populations for 1980, 1970, and 1960 are also included on the Population Data Sheet. Using either the "Bigger Picture" or the "Smaller Picture" method, determine a population center for one of these years. Compare your center to the center plotted on the map presented at the beginning of this lesson.

17. Some "futurists" speculate on what the population of countries in North America will look like 50 to 100 years from now. Two extreme scenarios were developed. One scenario suggests that the population center of the United States will be located close to the Canadian border. Another scenario suggests the population center will be located in the southwestern corner of the United States. Explain what would cause a population center to be located in either of these two extremes.

18. Examine a globe of the world. If you were interested in determining the population center of the world, what new questions must be addressed? Do you think a population center for the world exists? Is it meaningful? Explain.

Population Data Sets of the United States
Compiled from the *Statistical Abstract of the United States* (1991)

State	Population 1960 (in thousands)	Population 1970 (in thousands)	Population 1980 (in thousands)	Population 1990 (in thousands)
AK	226	303	402	550
AL	3267	3444	3894	4041
AR	1786	1923	2286	2351
AZ	1302	1775	2718	3665
CA	15717	19971	23668	29760
CO	1754	2210	2890	3294
CT	2535	3032	3108	3287
DE	446	548	594	666
FL	4952	6791	9746	12938
GA	3943	4588	5463	6478
HI	633	770	965	1108
IA	2758	2825	2914	2777
ID	667	713	944	1007
IL	10081	11110	11494	11431
IN	4662	5195	5490	5544
KS	2179	2249	2364	2478
KY	3038	3221	3661	3685
LA	3257	3645	4206	4220
MA	5149	5689	5737	6016
MD	3101	3924	4217	4781
ME	969	994	1125	1228
MI	7823	8882	9262	9295
MN	3414	3806	4076	4375
MO	4320	4678	4917	5117
MS	2178	2217	2521	2573
MT	647	694	787	799
NC	4556	5084	5882	6629
ND	632	618	653	639
NE	1411	1485	1570	1578
NH	607	738	921	1109
NJ	6067	7171	7365	7730
NM	951	1017	1303	1515
NV	285	489	800	1202

NY	16782	18241	17558	17990
OH	9706	10657	10761	10847
OK	2328	2559	3025	3146
OR	1769	2092	2633	2842
PA	11319	11801	11864	11882
RI	859	950	947	1003
SC	2383	2591	3122	3487
SD	681	666	691	696
TN	3567	3926	4591	4877
TX	9580	11199	14229	16987
UT	891	1059	1461	1723
VA	3967	4651	5347	6187
VT	390	445	511	563
WA	2853	3413	4132	4867
WI	3952	4418	4706	4992
WV	1860	1744	1950	1793
WY	330	332	470	454

Balancing the Map (Optional)

19. Consider estimating the population center with a map and eight raisins. Copy a map of the United States on poster paper.

Step 1: Cut out the map of the 48 connected states. Cut along the coast lines and the Canadian and Mexican borders.

Step 2: Tape eight raisins to eight estimated population centers for the country. Let each raisin represent approximately 30 million people. Based on the populations provided for each state, tape the eight raisins at the locations you think represent a population center for approximately 30 million people.

Step 3: Use the blunt end of the pencil discussed in previous lessons to balance your shape with the raisins.

Step 4: Find a balance point and record it. How does the balance point compare to the points calculated from the table?

Minimizing Distances by a Center

The floor plans of many buildings specifically consider the need of people to efficiently interact and move throughout the building. Architects of school buildings especially examine "traffic patterns" and efficient movement of people from one area of the building to other areas.

Survey your school building. Find an example of a location that might have been selected to "minimize the distance" people would need to walk in order to get to this location. In addition, develop your own sketch of a one-story school floor plan that considers:

- **the need for students to access a computer lab**
- **the need for teachers to meet with guidance counselors**

Why do you think your sketch meets the needs expressed above? Explain.

LESSON 11

Minimizing Distances on a Number Line

What are some of the things a person should consider when choosing a location for a distribution center that trucks supplies to several different stores?

Does the number of stores supplied by the warehouse matter?

Suppose it were your job to load a truck with supplies at a warehouse, drive the truck to a store, unload it, drive back to the warehouse, reload the truck with more supplies, drive it to a second store, unload it, and return to the warehouse. If you could choose the site for the warehouse, where would you put it?

The location of a center in this problem is different from location of a center in the problems involving a balance point or a population center. You will undoubtedly want to put the warehouse in a central location, but what is meant by "central" in this case? The goal is to find a location that minimizes the total distances driven to the stores and back.

OBJECTIVES

Determine the location along a number line that minimizes the sum of the distances to selected points.

Use the median of a set of points as an estimate for selecting a point to minimize the sum of distances.

INVESTIGATE

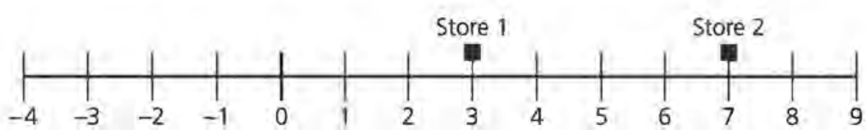
Location, Location, Location

Call the distance from the warehouse (no matter where it is located) to store 1 in the diagram d_1 and the distance from the warehouse to store 2 d_2. The total distance traveled by the truck in this problem is $2d_1 + 2d_2$. Why?

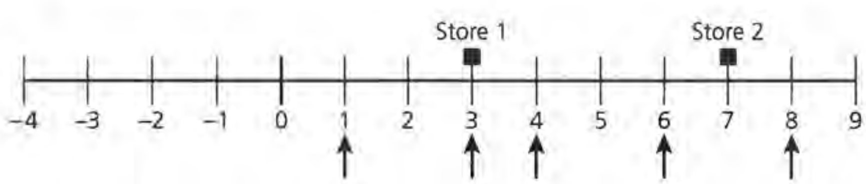
The goal in this problem is to *minimize* this *sum of the distances* $2d_1 + 2d_2$. This problem can be simplified by first minimizing $d_1 + d_2$. (The location of the warehouse that minimizes the sum of twice each distance is the same as the location that minimizes the sum of the distances—it does not matter if several one-way trips or several round trips are considered in this problem.)

Discussion and Practice

To find this new "center," visualize the setup of this problem on a number line (similar to the way the previous investigation of a center of balance was started). Each store will be placed on the number line. The goal of the problem is to determine the location of a warehouse that minimizes the sum of the distances discussed in this problem. To start this discussion, suppose the stores are located along a straight road in which the location of store 1 is 3 and the location of store 2 is 7.



1. The best location for the warehouse is somewhere along the line connecting the two stores. Why?
2. What would be your initial estimate for the location of a warehouse that minimizes the sum discussed in this lesson? Why did you choose that location?
3. Consider locating the warehouse at each of the locations designated in the following chart. Determine the sum of the one-way distances to each of the stores based on the warehouse location:



Location of Warehouse	Distance to Store 1	Distance to Store 2	Sum of the Distances
Example:			
8	5	1	6
6	______	______	______
4	______	______	______
3	______	______	______
1	______	______	______

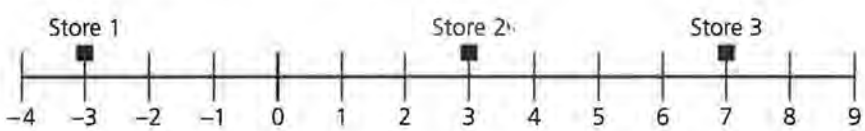
4. Consider the results from the chart.

a. What location or locations minimize the sum of the distances?

b. How would you describe the possible locations of the warehouse?

5. Would the location that minimizes the sum of the distances for the two stores be the same location if the center of balance were used to locate the warehouse. Why or why not?

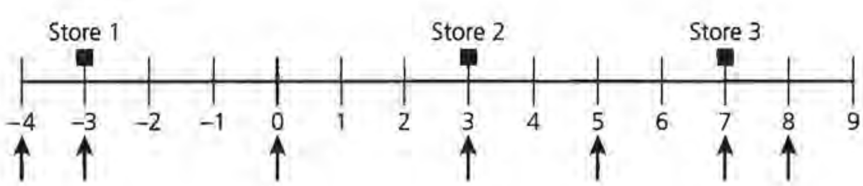
The location of the warehouse for this first example is simplified by the fact that only two stores are considered. Consider an expansion to this problem by placing a third store at location –3 on this number line.



The location of the warehouse must minimize the sum of the distances:

distance from warehouse to Store 1
+ distance from warehouse to Store 2
+ distance from warehouse to Store 3.

6. Consider several possible locations to place this warehouse. Each of the possible locations are identified in the following diagram.



a. Copy and complete the chart summarizing the distances discussed in this problem:

Location of Warehouse	Distance to Store 1	Distance to Store 2	Distance to Store 3	Sum of the Distances
Example:				
−4	1	7	11	19
−3	_____	_____	_____	_____
0	_____	_____	_____	_____
3	_____	_____	_____	_____
5	_____	_____	_____	_____
7	_____	_____	_____	_____
8	_____	_____	_____	_____

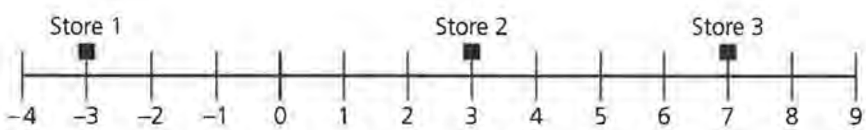
b. Would you have added any other possible locations of the warehouse to this chart? If yes, determine the sum of the distances for the additional locations you considered.

7. Based on the information summarized by the table in question 6, what location or locations would minimize the sum of the distances?

8. How would you describe the location of the warehouse that minimizes the sum of the distances?

9. Would the location that minimizes the sum of the distances for three stores be the same location if the center of balance were used to locate the warehouse? Why or why not?

10. Suppose store 3 had been at location 9, rather than at location 7.



a. Copy and complete the chart summarizing the distances:

Location of Warehouse	Distance to Store 1	Distance to Store 2	Distance to Store 3	Sum of the Distances
–3	______	______	______	______
0	______	______	______	______
3	______	______	______	______
5	______	______	______	______
7	______	______	______	______
9	______	______	______	______

b. Would you have added any other possible locations of the warehouse to this chart? If yes, determine the sum of the distances for the additional locations you considered.

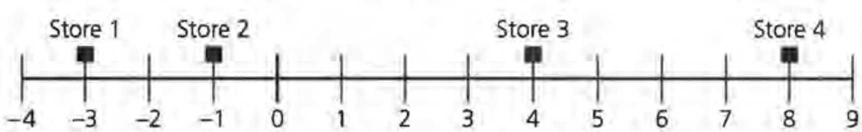
c. What is the best location for the warehouse? In what way, if any, does moving store 3 from location 7 to location 9 affect the best location for the warehouse? Why?

SUMMARY

Different types of centers are based on different types of problems. A central location designed to minimize the sum of the distances to selected points along a number line is estimated by the location of the median of a set of points. Depending on the number of points described in the problem, the median may not necessarily be the only location that minimizes the sum of the distances, however, it is the estimate that begins the investigation of this location.

Practice and Applications

11. Consider four stores located along a number line as illustrated in the following diagram:



a. Based on your work with two and three stores, where would you suggest a warehouse be located to minimize the sum of the distances?

b. Determine the sum of the distances for at least five possible locations of the warehouse. Include in your locations the one you think is the best. Organize the results of each location in a chart similar to the one below.

Location of Warehouse	Distance to Store 1	Distance to Store 2	Distance to Store 3	Distance to Store 4	Sum of the Distances
____	____	____	____	____	____
____	____	____	____	____	____
____	____	____	____	____	____
____	____	____	____	____	____
____	____	____	____	____	____
____	____	____	____	____	____

12. Describe the best location or locations of the warehouse for four stores located along a number line.

13. If it were necessary to locate the warehouse at the location of an actual store, what store or stores would you choose in problem 12? Why?

14. Why is it more difficult selecting the location of a warehouse for four stores than for three stores?

15. Draw a number line and place the location of five stores along the number line.

a. Where would you locate a warehouse to minimize the sum of the distances to the stores?

b. Is there more than one possible location for a warehouse when there are five stores? Why or why not?

LESSON 12

Taxicab Geometry

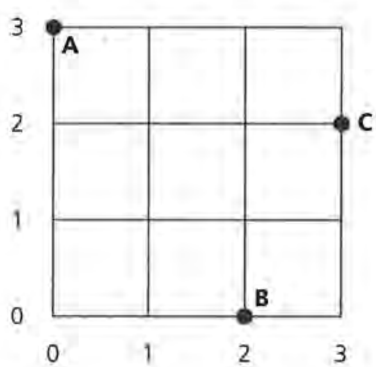
A warehouse is to be located so that it can supply three stores, labeled A, B, and C in the diagram below. If the warehouse were to be added to an existing store, which store would you select?

What other in-town location would be a better site? Why?

Suppose there are three stores located at the intersections of streets in a town. A street map of the town shown below looks like a grid. All of the streets run either north-south or east-west and are evenly spaced. Locations of the stores can be designated by using (x, y) coordinates. For example, store B is located on this map as location (2, 0). What is the location of store A?

OBJECTIVE

Determine the location on a two-dimensional grid that minimizes the sum of the distances to selected points.



INVESTIGATE

Who's Minding the Store?

Where is the best place to put a "central" warehouse? The goal is to *minimize* the sum of the distances from the warehouse to the stores. Measure this distance by counting the number of blocks traveled.

Discussion and Practice

1. What is the distance from store B to store C in the diagram?

2. Store C could be moved to a different location and remain the same distance from store B as it is now. Could store A be moved to a different location on the grid and have it be the same distance from store B as it is now?

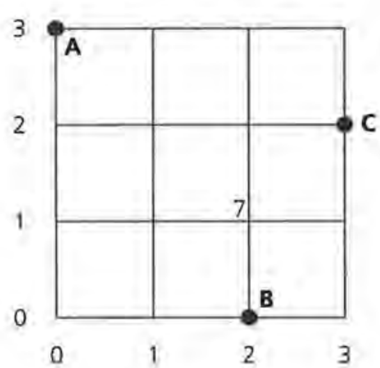
3. What is your initial estimate for the location of a warehouse that minimizes the sum of distances to the three stores? Why did you choose that location?

4. Consider putting the warehouse discussed in problem 3 at the location with coordinates (2, 1). The distance from this warehouse to store A is 4 blocks. Determine the distance from the warehouse to store B and from the warehouse to store C.

5. Consider locating the warehouse at each of the locations designated in the following chart. Find the distances needed to fill in the table and find the sum of the distances.

Location of Warehouse	Distance to Store A	Distance to Store B	Distance to Store C	Sum of the Distances
Example:				
(2, 1)	4	____	____	7
(3, 1)	____	____	____	____
(2, 2)	____	____	____	____
(2, 3)	____	____	____	____

If the warehouse is located at (2, 1), the sum of the distances is 7 blocks. Label location (2, 1) with a "7" on the map to designate this location.

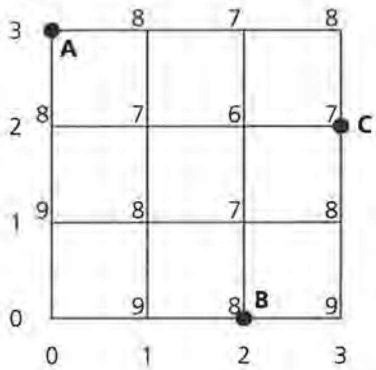


6. Consider locating the warehouse at other locations.

   a. If the warehouse is located at (3, 1), what is the sum of the distances to the three stores?

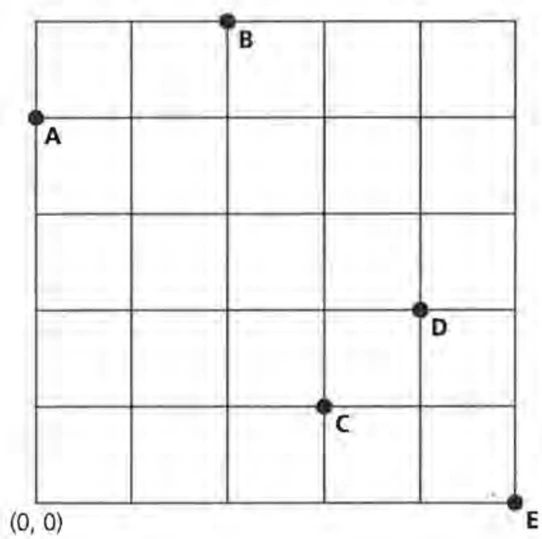
   b. If the warehouse is located at (2, 2), what is the sum of the distances?

Placing a number representing the sum of the distances at each coordinate location gives the following summary:



7. According to this summary, what location minimizes the sum of the distances?

Expand this problem to include five stores located as shown on the grid.



The question remains to determine the location of a warehouse that minimizes the sum of the distances to each store. One possible location for the warehouse is (4, 1), included on the following table. Another possible location is (4, 2). This location indicates the possibility of locating the warehouse at the same location as store D.

8. In addition to the locations of (4, 1) and (4, 2), choose four other possible locations for the warehouse. For each local, calculate the sum of the distances to the stores. Copy and complete the following table for each of the choices:

Location of Warehouse	Distance to Store A	Distance to Store B	Distance to Store C	Distance to Store D	Distance to Store E	Sum of the Distances
Example:						
(4, 1)	7	6	1	1	2	17
(4, 2)	6	5	2	0	3	16
______	______	______	______	______	______	______
______	______	______	______	______	______	______
______	______	______	______	______	______	______
______	______	______	______	______	______	______

9. What is the best location for the warehouse as represented with the above locations?

10. What is the x-coordinate of the best location for the warehouse? How is this related to the x-coordinates of the locations of the five stores?

11. What is the y-coordinate of the best location for the warehouse? How is this related to the y-coordinates of the locations of the five stores?

12. Based on the solutions to problems 10 and 11, how would you describe the location for the warehouse that minimizes the sum of the distances?

SUMMARY

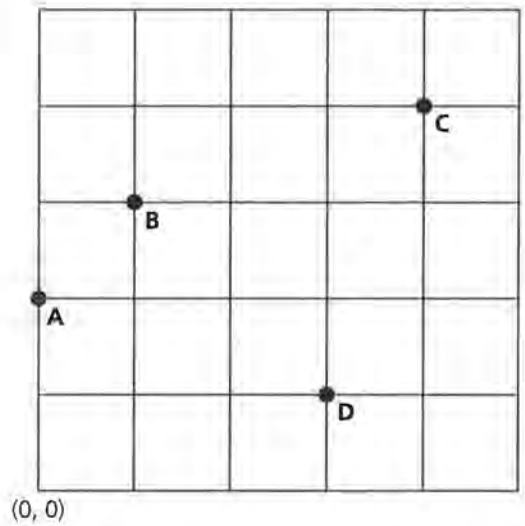
The location that minimizes the sum of distances to selected points along a grid is given by the **median** of the x-coordinates and the **median** of the y-coordinates. Depending on the number of points in the problem, there may be more than one location that minimizes the sum of the distances.

Practice and Applications

13. Suppose there are four stores, located as on the grid below.

a. Based on your work with three stores and with five stores, what is the best location for the warehouse? Do you think there might be more than one location that is "best"?

b. Choose possible locations for the warehouse, calculate the sum of the distances for each location, and fill in these values on the table.



Location of Warehouse	Distance to Store A	Distance to Store B	Distance to Store C	Distance to Store D	Sum of the Distances
______	______	______	______	______	______
______	______	______	______	______	______
______	______	______	______	______	______
______	______	______	______	______	______

14. Consider the first example (with three stores) again. Suppose that you need to go from the warehouse to store A once, from the warehouse to store B once, but from the warehouse to store C three times. How does this change the problem? Now where is the best place for the warehouse?

Helicopter Geometry

What would be the advantages of using a helicopter instead of a truck to transport supplies from a distribution center to area stores?

What would the disadvantages be?

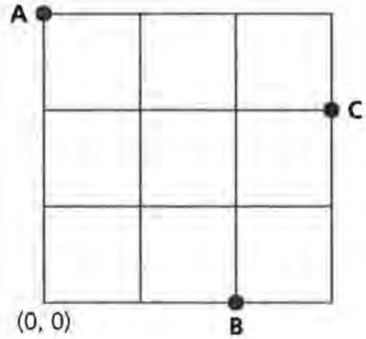
OBJECTIVE

Determine the location on the plane that minimizes the sum of distances to selected points.

In Lesson 12, the best location for a warehouse that served a group of stores was found under the condition that travel was restricted along gridlines. The "center" turned out to involve the medians of the x- and y-coordinates of the stores. Suppose there is no restriction that travel must be along gridlines. Instead, suppose that you have a helicopter and can travel on a direct line to each point. It turns out that this makes the solution—the best location for the warehouse—harder to find!

INVESTIGATE

Consider three stores given by the points on the map shown below. The goal of the next set of problems is to find the best location for the warehouse, given that you can fly directly from the warehouse to each point. In other, words, *the warehouse does not have to be at the intersection of streets any more!* Use the gridlines as a kind of graph paper to help keep track of locations on the map.



Discussion and Practice

If the warehouse is placed at (0, 0), then the distance from the warehouse to point A is 3 units.

1. What is the distance from (0, 0) to point B?
2. Use the Pythagorean Theorem to verify that the distance from (0, 0) to point C is $\sqrt{3^2+2^2} = \sqrt{13}$ units.
3. What is your initial estimate for the location of a warehouse that minimizes the sum of distances to the three stores? Why did you choose that location?

Consider Three Points

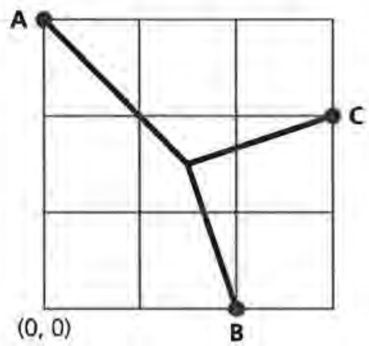
If the warehouse is at location (0, 0), then the sum of the distances is $\sqrt{3^2+2^2} = \sqrt{13}$, or about 3.6.

4. Locate the warehouse at location (1, 1). The distance from the warehouse to store A is $\sqrt{5}$ units (again, derived by way of the Pythagorean Theorem). Determine the distance from the warehouse to store B and from the warehouse to store C.
5. Consider locating the warehouse at each of the locations designated in the following chart. Find the distances needed to complete the table and find the sum of the distances.

Location of Warehouse	Distance to Store A	Distance to Store B	Distance to Store C	Sum of the Distances (rounded to one decimal place)
(0, 0)	3	______	$\sqrt{13}$	8.6
(1, 1)	$\sqrt{5}$	______	______	______
(1, 2)	______	______	______	______
(2, 2)	______	______	______	______

Before a decision can be made concerning the location of the warehouse, locations that are not at the intersection of gridlines must also be considered. For example, suppose the warehouse is located at (1.5, 1.5). One way to estimate the distances involved is to measure the distance to each point representing a store with a piece of string. Anchor one end of the string at one of the possible locations of the warehouse; stretch the string and mark the location to store A. Again, anchor the string at the warehouse location and mark the distance to store B. Finally, anchor the string and mark the distance to store C. The accumulated distances of the string represents the sum of the distances requested in this problem.

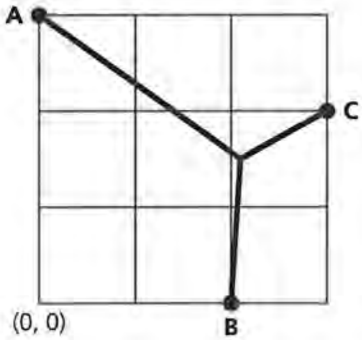
In order to improve these measurements, plot the stores A, B, and C, on the larger grid provided with this material. Express the final distance of the string in terms of the units of this grid. For example, if you place the warehouse at (1.5, 1.5), then the distance to point A is estimated at 2.1 units; similarly, the distance to B is 1.6 units, and the distance to C is 1.6 units. The sum of the distances is approximately 5.3 units. A more accurate method to investigate this problem is to use one of several software packages such as The Geometer's SketchPad or Cabri. Each application allows you to locate the stores on a grid and develop a sum of the distances.



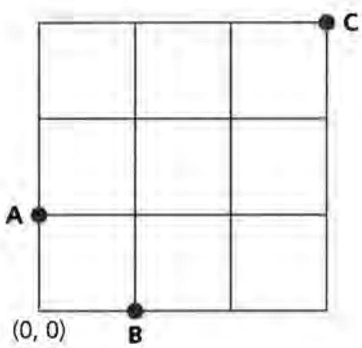
6. Consider locating the warehouse at each of the locations designated in the following chart. Use a piece of string to find the distances to each of the stores and record those in the table. Then find the sum of the distances.

Location of Warehouse	Distance to Store A	Distance to Store B	Distance to Store C	Sum of the Distances (rounded to one decimal place)
(1.5, 1.5)	2.1 units	1.6 units	1.6 units	5.3 units
(1.5, 2)	______	______	______	______
(2.5, 2.5)	______	______	______	______

The best location for the warehouse given the locations of the stores is at (2.1, 1.6); this is illustrated in the following graph. The sum of the distances in this case is approximately 5.1 units, the smallest value possible.



7. Suppose the three stores are at points (0, 1), (1, 0), and (3, 3), as shown in the following diagram. Estimate where the location of the warehouse should be. Explain your choice.

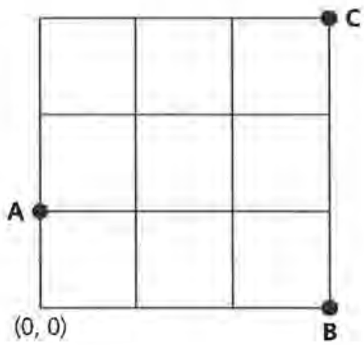


8. Using the larger grid provided with this module, plot the location of the three stores as indicated in the previous diagram. Select any three points on the grid to represent the location of the warehouse. Use a piece of string or a computer program to find the distances to each of the stores. Record your results in the following table. Find the sum of the distances.

Location of Warehouse	Distance to Store A	Distance to Store B	Distance to Store C	Sum of the Distances (rounded to one decimal place)
______	______	______	______	______
______	______	______	______	______
______	______	______	______	______

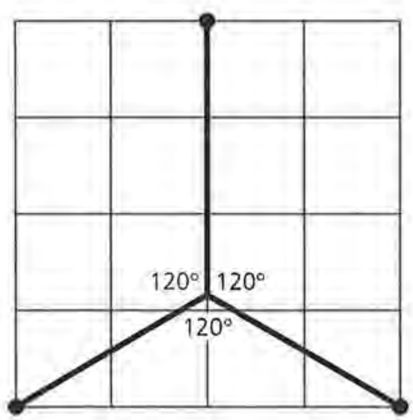
9. The best location for the warehouse gives a sum of distances of 4.7 units. Can you find that location?

10. Suppose three stores are at points (0, 1), (3, 0), and (3, 3), as shown below. Where should the warehouse be for this arrangement? Explain your estimate.



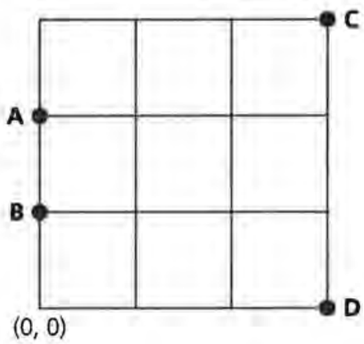
11. On the basis of the preceding three examples, what can you say about how the best location for the warehouse is related to the locations of the three stores?

You may have noticed that when you have found the best location for the warehouse and you draw lines between the warehouse and each store, those lines are at 120-degree angles from one another.



Consider Four Points

Suppose there are four stores, rather than three. For example, consider the following configuration of stores A, B, C, and D:



Where should the warehouse be located?

12. What is your initial estimate for the location of a warehouse that minimizes the sum of distances to the four stores? Why did you choose that location?

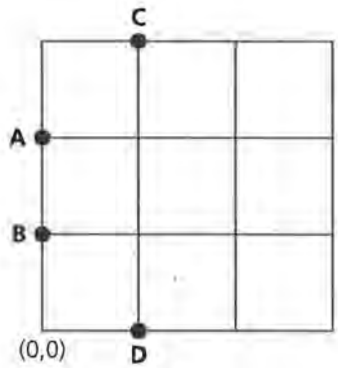
If the warehouse is at location (1, 1), then the sum of the distances is $\sqrt{2} + 1 + \sqrt{8} + \sqrt{5}$, or about 7.5.

13. Consider locating the warehouse at each of the locations designated in the following chart. Determine the distances required to complete this table by string or computer application. (If estimates are made by way of string, transfer the points to the larger grid provided with this material.)

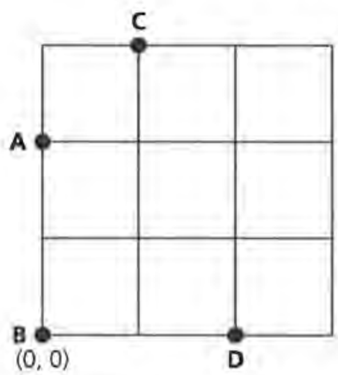
Location of Warehouse	Distance to Store A	Distance to Store B	Distance to Store C	Distance to Store D	Sum of the Distances (rounded to one decimal place)
(2, 2)	______	______	______	______	______
(1, 1.5)	______	______	______	______	______
(.5, 1.5)	______	______	______	______	______
(.75, 1.5)	______	______	______	______	______

14. What is the best location for the warehouse?

15. Suppose the four stores are at points (0, 2), (0, 1), (1, 3), and (1, 0), as shown below. Now where should the warehouse be?



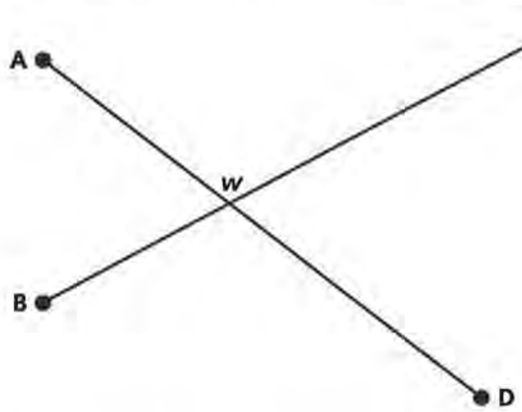
16. Suppose the four stores are at points (0, 2), (0, 0), (1, 3), and (2, 0), as shown below. Where should the warehouse be?



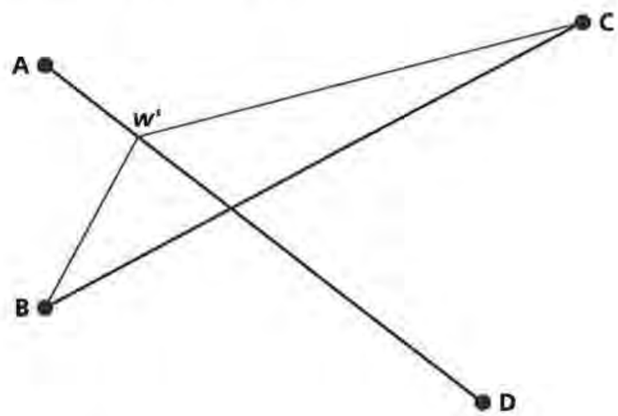
17. On the basis of the preceding three examples, what can you say about how the best location for the warehouse is related to the locations of the four stores?

You may have noticed that the best location for the warehouse is at the intersection of the line segments formed by connecting each point to the point "opposite" it.

To see that this is the best possible location, consider the following. Suppose the warehouse were located at the point of intersection, labeled as w in the following diagram:



Now suppose it is suggested to move the warehouse closer to point A by sliding along the diagonal between A and D. Call this new location w'.

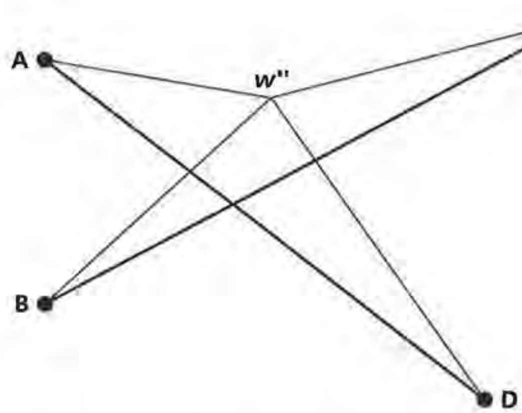


18. How does moving the warehouse from location w to location w' affect the sum of the distances from the warehouse to A and to D? Why?

19. How does moving the warehouse from location w to location w' affect the sum of the distances from the warehouse to B and to C? Why? (Hint: consider the triangle inequality.)

20. Combine your answers to problems 18 and 19 to show that w is a better location than w'.

21. Suppose the warehouse were moved to a spot such as w'' in the following diagram, which is not on either diagonal. Explain how the triangle inequality can be applied twice to show that w'' is a worse location than w.



Consider Five or More Points

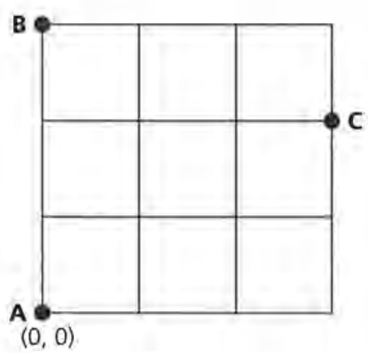
If there are five or more points, then there is no simple method to find the best location for the warehouse. Use string and trial and error to find the best location.

SUMMARY

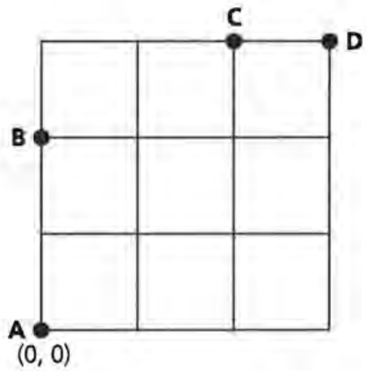
When there are three stores, the location that minimizes the sum of distances to the stores is given by the spot such that line segments from the stores to the spot meet at 120-degree angles. When there are four points, the location that minimizes the sum of the distances is the intersection of line segments connecting pairs of opposite points. When there are five or more points, then there is no simple method for finding the best location.

Practice and Applications

22. Suppose there are three stores and they are at points (0, 0), (0, 3), and (3, 2), as shown below. Where should the warehouse be?



23. Suppose there are four stores and they are at points (0, 0), (0, 2), (2, 3), and (3, 3), as shown on the following diagram. What is the best location of the warehouse?



LESSON 14

The Worst-Case Scenario!

Jacksonville, Florida, is a "port of entry," a U.S. town that receives cars imported from other countries. Would you expect the price of a new car in Jacksonville to be the same as the price of the same model car in Minneapolis?

You have agreed to buy a new car, but the dealer currently does not have the car in the color you want. How does the dealer find and get the car you want?

OBJECTIVE

Calculate the value that minimizes the greatest distance.

Reconsider the warehouse problem presented in Lesson 11. Suppose that instead of stocking every store at the end of the week, a different delivery problem is presented. In this new situation, stores call the warehouse when a sale has been made; it is the company's goal to deliver that product from the central warehouse to the store as soon as possible. For example, car dealerships cannot stock every model in every color combination. Once a customer has selected a car with specific features and in a certain color, someone from the central storage warehouse must drive the customer's selection to the dealership. If this is your job, you hope the dealership is close to the "central" warehouse. On some unlucky days, you will be required to travel to the dealership that is farthest from the central warehouse. Thus, the car company considers locating the central warehouse so that *the distance to the farthest dealership is minimized.*

INVESTIGATE

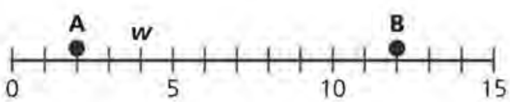
New Car Delivery

This problem and its solution are different from the problem and solution presented in Lesson 11. In Lesson 11, the goal was to minimize the sum of the distances to the stores. The goal in the problems of this lesson is to minimize the *greatest* distance.

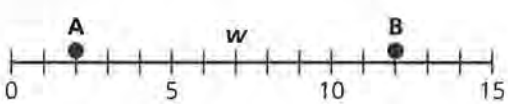
As in previous problems, suppose first that you are concerned with only two stores, A and B, that are at locations 2 and 12 on a number line. A warehouse is to be built between these two stores so that the greatest distance to a store is minimized.

Discussion and Practice

1. What is your initial estimate of the location that minimizes the greatest distance? Why did you choose that location?
2. Suppose the warehouse is located on the number line at $w = 4$. Use the number line that follows to answer the following questions.
   a. What is the distance from w to A?
   b. What is the distance from w to B?
   c. Is the distance greater from the warehouse to store A or to store B?



3. Suppose the warehouse is located at $w = 7$. What is the greater distance from the warehouse to a store?



Which Store is Farther?

Consider walking along the number line. Wherever you are located, you can stop and ask "Which store is farther from here?" If you are to the left of 7, then the answer to this question is "Store B"; if you are to the right of 7, then the answer is "Store A." Putting the warehouse at $w = 7$ is the best we can do; if we move the warehouse away from 7, then either the distance to A gets larger than 5 or the distance to B gets larger than 5.

4. For each of the following, determine the location of the point that minimizes the greater distance:

   a. Store A is located at -4 and store B is located at 6.

   b. Store A is located at -8 and store B is located at -1.

   c. Store A is located at -6.5 and store B is located at 2.

Now consider the problem in which there are several stores on a line. Consider stores A, B, and C at locations 2, 12, and 5 on a number line.

5. What is your initial estimate of the location that minimizes the greatest distance? Why did you choose that location?

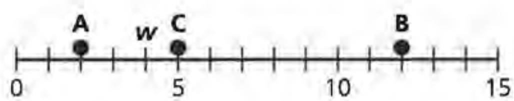
6. Suppose the warehouse is located at $w = 4$.

   a. What is the distance from w to A?

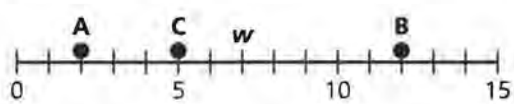
   b. What is the distance from w to B?

   c. What is the distance from w to C?

   d. What is the greatest distance from the warehouse to a store?



7. Suppose the warehouse is located at $w = 7$. What is the greatest distance from the warehouse to a store?



Again, putting the warehouse at $w = 7$ is the best location to answer this question; if the warehouse is moved away from 7, then either the distance to A gets larger than 5 or the distance to B gets larger than 5.

8. Why does the location of C not enter into the consideration of the warehouse's location. (Hint: consider walking along the number line asking yourself, "Which store is farthest away from me?")

What happens if there are n stores on a number line? Consider stores $P_1, P_2, \ldots, P_n$ on a number line.

As you walk along this number line, there are only two possible answers to your favorite question, "Which store is farthest from me?" Sometimes store P_1 is the most distant store and sometimes store P_n is the most distant store. If you start to the left of P_1 and walk toward the right, then P_n is the most distant store for the first part of the walk. Eventually P_1 becomes the most distant point. Where does this switch happen?

The ***midrange*** is defined to be:

$$\frac{\text{Minimum value} + \text{Maximum value}}{2}$$

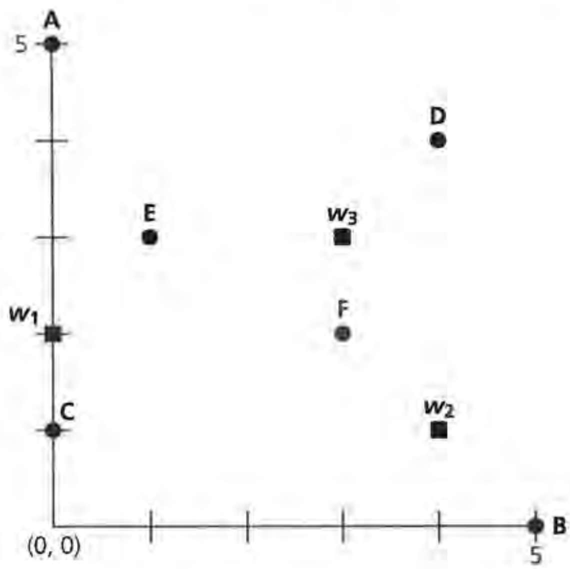
This is another type of center and is the solution to the problem for points located on a number line.

9. Use the midrange to determine the location on a number line of the points requested in the following statements:

a. Determine the location of point w that minimizes the greatest distance from point w to any one of the following stores: store A at –5, store B at –2, store C at 3, and store D at 8.

b. The location of the central distribution warehouse for four car dealerships (namely A, B, C, and D) is at the location of 4 on a number line. Store A is located at –1, store B at 2 , and store C at 5. Determine the location of store D. Explain how you determined your solution to this statement.

Now consider a more realistic setting in which the locations of the stores do not fit on a simple number line. A more appropriate representation of the stores is a set of points on a two-dimensional map. Consider the representation of six stores, A–F, as illustrated on an xy-coordinate system:



The graph above shows three locations that are being considered for the central warehouse: w_1, w_2, and w_3. The decision of which location to select will be based on the goal of minimizing the greatest distance traveled to any of the stores.

10. Which location do you think should be selected? Why?

11. Collect the information needed to make this decision. Hold a piece of string at location w_1. Measure the length of string to store A and record this length on the summary sheet. Similarly, determine the length of string from w_1 to stores B, C, D, E, and F. Repeat this process by holding the string at w_2 and recording the length to each of the six stores. Also, collect the data using location w_3 as the location of the central warehouse.

Summary Sheet

Coordinate Location of Store (x, y)	Distance of Store from w_1	Distance of Store from w_2	Distance of Store from w_3
A	______	______	______
B	______	______	______
C	______	______	______
D	______	______	______
E	______	______	______
F	______	______	______

From the above data, determine the following:

Central Warehouse	Store Farthest from Central Warehouse	Distance from Farthest Store to Central Warehouse
w_1	______	______
w_2	______	______
w_3	______	______

12. From the summary data collected, determine which suggested warehouse location minimizes the distance to the farthest store.

13. What if you were not given the suggested points w_1, w_2, and w_3? Can you find a location that is better than any of these three?

14. Suppose that no restraints are placed on the location of the central warehouse.

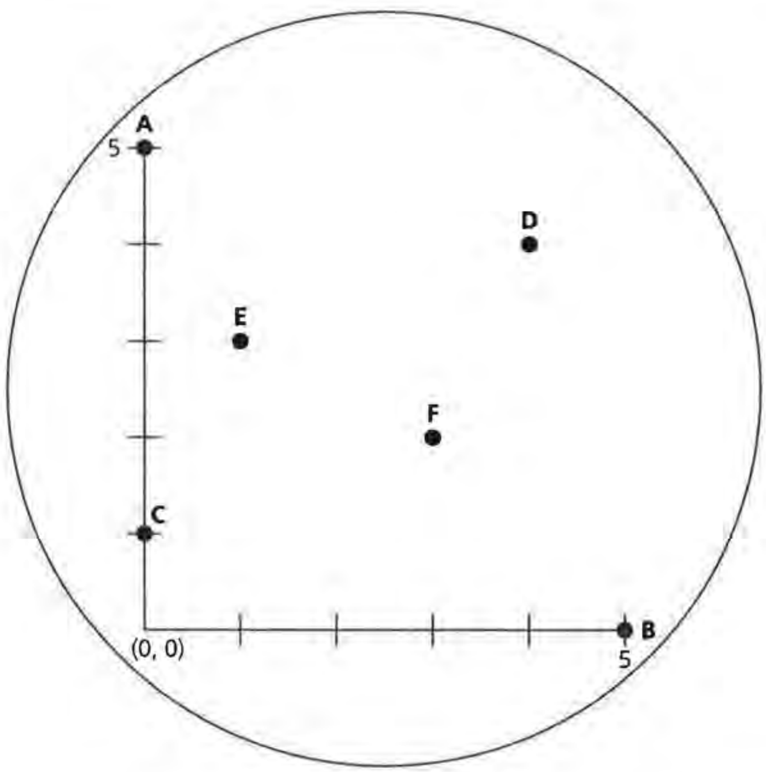
  a. With a compass, construct a circle that encompasses all six of the points on the map. Construct this circle on a sheet of paper that can be placed over the map points. Your paper should be somewhat transparent so that the map points show through. Remember, it is the center of this circle that matters; therefore, record the center you used to construct this first circle.

  b. Using the center of this circle, continue to construct smaller circles that encompass the map points. Repeat this until one of your **shrunken** circles touches a point on the map.

c. Now move or **slide** this circle so that all six of the points are enclosed in the circle. Record the new center of the circle you have discovered.

d. Continue experimenting with smaller and smaller circles at this location until the circle touches at least one point. Then slide the circle so that all points are inside this circle.

e. Eventually, you will be no longer able to "**shrink or slide**" the circle and still have it encompass all six points. If you shrink this circle any further, at least one of the points of the map will be located outside of the circle. The **center** of this circle **minimizes the greatest distance.** Why?



f. Complete the chart below. Which store is farthest from the center, w, of your circle?

Center w (x, y)	Length to Store A	Length to Store B	Length to Store C	Length to Store D	Length to Store E	Length to Store F
______	______	______	______	______	______	______

SUMMARY

When dealing with points on a number line, the location that minimizes the greatest distance to any of the points is the *midrange* of the values. When dealing with points on a plane, the location that minimizes the greatest distance to any of the points is the center of the smallest circle that encompasses all of the points.

Practice and Applications

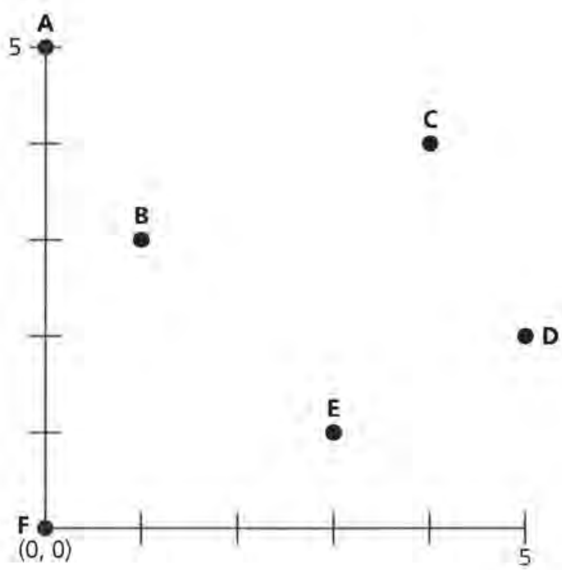
15. For each of the following, develop a number line and plot the specific points. Use the midrange to determine the location on your number line of the points requested in the following statements:

   a. Determine the location of the point that minimizes the greatest distance to any one of the following stores: store A at 2, store B at 5, and store C at 7.

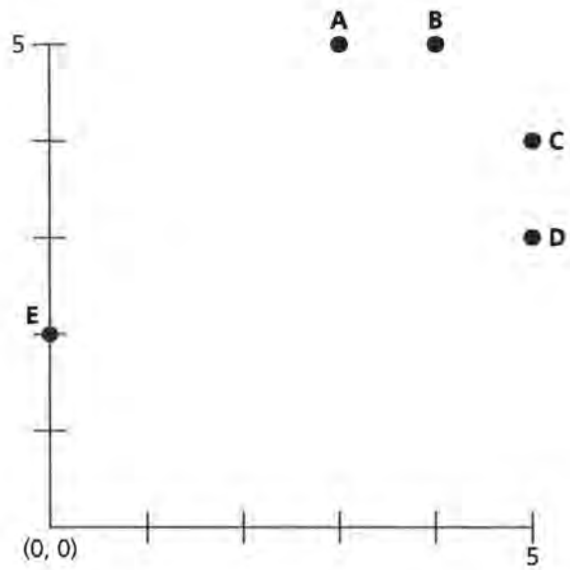
   b. Determine the location of the point that minimizes the greatest distance to any one of the following stores: store A at –3, store B at –1, store C at 2, store D at 11, and store E at 15.

Consider the location of stores for each of the situations plotted on the following *xy*-coordinate systems. Using the "shrink and slide" ideas, approximate the center point that minimizes the location from the center to the point farthest from the center. If possible, develop these problems using a geometric software program like Sketchpad or Cabri.

16.



17.



18. For this problem, create a map similar to the type presented in problems 16 and 17. Design your map for at least five stores that have a *center* at the location of (3, 2) on an *xy*-coordinate system.

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