Nonparametric Estimation of First Passage Time Distributions in Flowgraph Models

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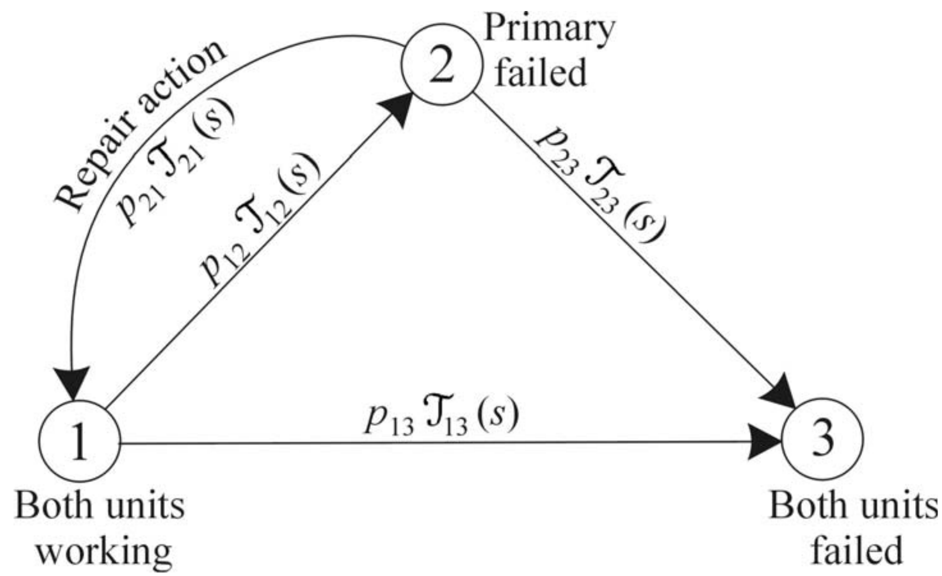
Outline

- Multistate models
- Overview of parametric statistical flowgraphs
- Nonparametric flowgraph models
  – Empirical transforms
- Reliability applications
  – Repairable redundant system
  - Competing risk of wearout/random failures
  - Detecting a model error
- Pros & cons of nonparametric method
- Summary and conclusions

Multistate model example

- Repairable system with redundancy
  - Or . . . reversible illness-death model
- In general, models of complex systems with many failure modes, feedback (repair) loops

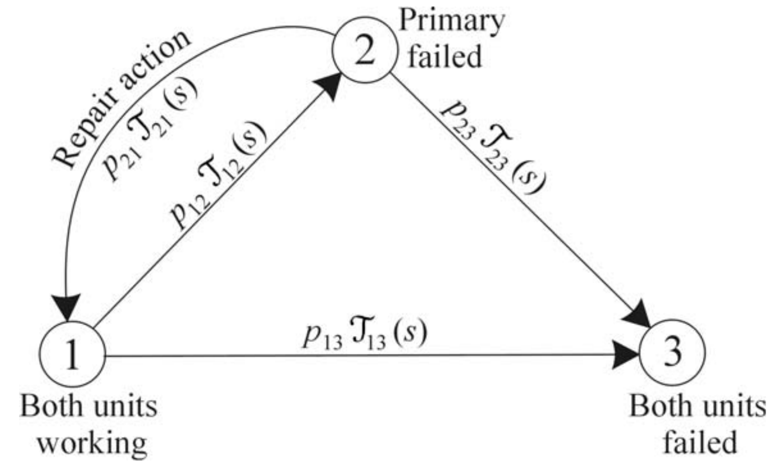


- Interested in time distribution for first passage from "working" to "failing" state – 1 $\rightarrow$ 3 in this example

Formally: time-homogeneous semi-Markov process

- Transition probabilities p_{ij} (embedded Markov chain)
- Arbitrary holding time distributions $F_{ij}(t)$

Overview of "classical" parametric flowgraphs



- <u>Transmittance</u> between states: $p_{ij}\,\mathfrak{T}_{ij}(t)$, $\mathfrak{T}_{ij}(t)$ is MGF or Laplace transform of $F_{ij}(t)$
- Parallel transmittance = sum, series transmittance = product, of branch transmittances
- Transforms reduce solution to an algebraic computation (Mason's rule), e.g., for 1→3 first passage:

$$\mathfrak{T}(s) = \left[p_{12}\,\mathfrak{T}_{12}(s)\,p_{21}\,\mathfrak{T}_{21}(s)\right]\mathfrak{T}(s) + p_{12}\,\mathfrak{T}_{12}(s)\,p_{23}\,\mathfrak{T}_{23}(s) + p_{13}\,\mathfrak{T}_{13}(s)$$

$$= \frac{p_{01}\,\mathfrak{T}_{01}(s)\,p_{12}\,\mathfrak{T}_{12}(s)}{1 - p_{12}\,\mathfrak{T}_{12}(s)\,p_{21}\,\mathfrak{T}_{21}(s)}$$

- Inversion of computed MGF/LT yields first passage density

A. V. Huzurbazar, *Flowgraph Models for Multistate Time-to-Event Data*, Wiley 2005.

Empirical transforms

<u>Exact transform</u> with kernel ψ of a distribution $F(t)$:

$$\Im(s) = E[\psi(t,s)] = \int \psi(t,s)\,dF(t)$$

Corresponding <u>empirical transform</u> for sample $\{t_1, \ldots t_n\}$ from F:

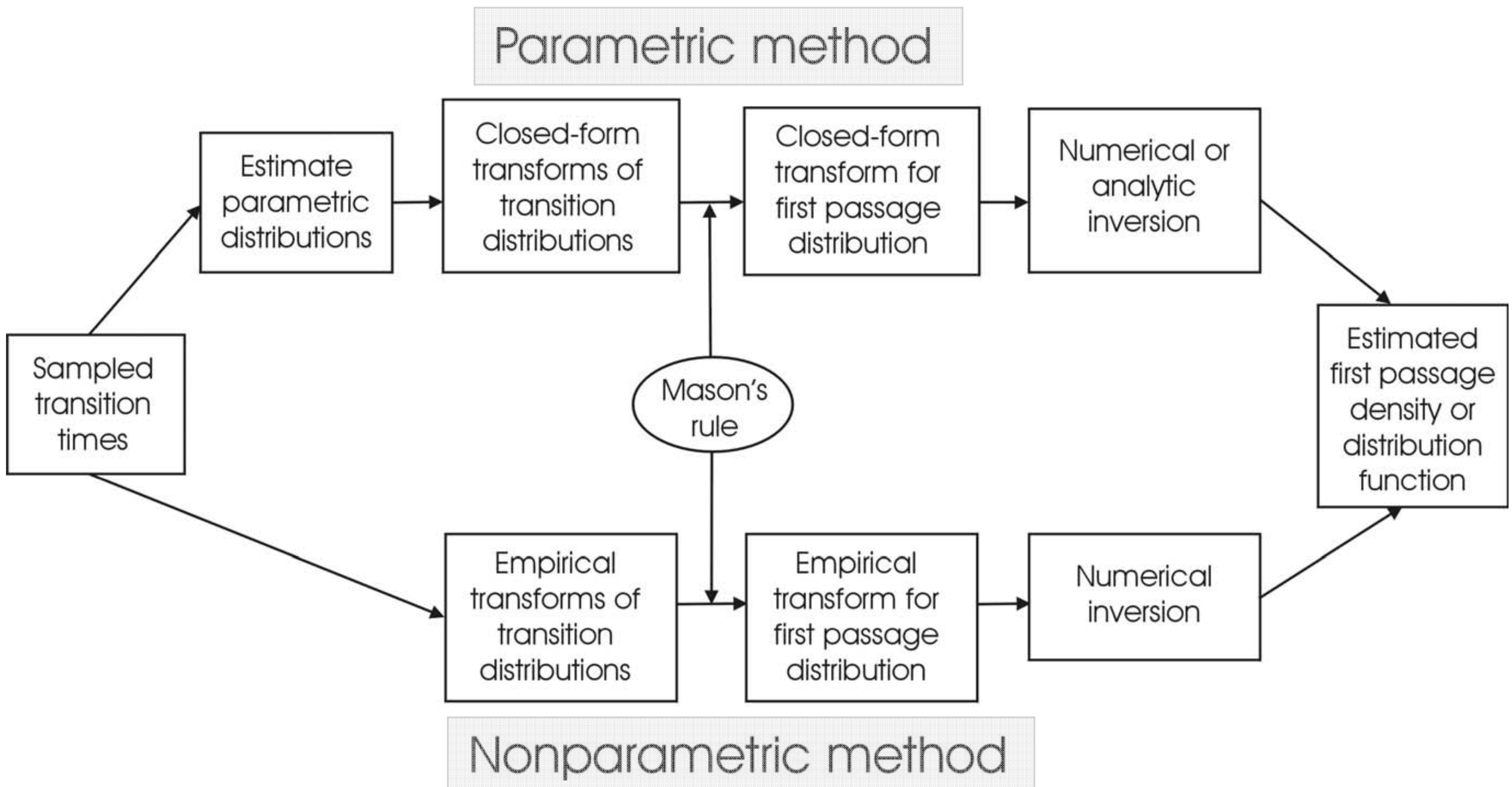
$$\tilde{\Im}(s) = \int \psi(t,s)\,dF_n(t) = \frac{1}{n}\sum_{i=1}^{n}\psi(t_i, s)$$

E.g., Laplace transform (f is the density of F):

$$\mathcal{L}_f(s) = \int_0^\infty e^{-st} f(t;\theta)\,dt, \qquad \tilde{\mathcal{L}}_f(s) = \frac{1}{n}\sum_{i=1}^{n} e^{-st_i}$$

- By SLLN, empirical converges to exact transform
- Retains standard properties, e.g., convolution

Parametric and nonparametric flowgraph methods



Empirical transforms (continued)

- Can mix and match exact and empirical transforms
- Empirical transform for right-censored data can be defined based on Kaplan-Meier estimator
- Inversion methods
  - Saddlepoint approximation based on MGF
  - Fourier series approximation of LT inversion
- Advantages/disadvantages of empirical transforms:
  - Harder to invert (computational complexity)
  - Less accurate then parametric **<u>if</u>** family is known
  - Can be **<u>more</u>** accurate – error is bounded by the data, not the model (see example later)

Numerical inversion of empirical transforms

Objective: recover the density from which a "pseudosample" of first passage times could be drawn

1. Saddlepoint method (Daniels 1954)

$$\hat{f}(t) = \left[\frac{1}{2\pi K''(\hat{s}_t)}\right]^{1/2} \exp\left[K(\hat{s}_t) - \hat{s}_t t\right]$$ where $\hat{s}_t$ (the saddlepoint)

solves $K'(\hat{s}_t) = t$, $\quad K(s) = \log\left[\mathcal{M}(s)\right]$ is the CGF.

Empirical saddlepoint substitutes $\tilde{K}(s) = \log\left[\frac{1}{n}\sum_{i=1}^{n} \exp(st_i)\right]$,

the empirical cgf

Daniels, H. E. (1954), "Saddlepoint approximations in statistics," *The Annals of Mathematical Statistics* 25, 631-650.

Numerical inversion of empirical transforms

2. Approximate the inversion integral with a truncated Fourier series:

$$s_N(t) = \frac{he^{at}}{\pi}\mathrm{Re}\left[\mathcal{L}_f\left(a\right)\right] + \frac{2he^{at}}{\pi}\sum_{k=1}^{N}\mathrm{Re}\left\{\mathcal{L}_f\left(a+ikh\right)\right\}\cos(kht)$$

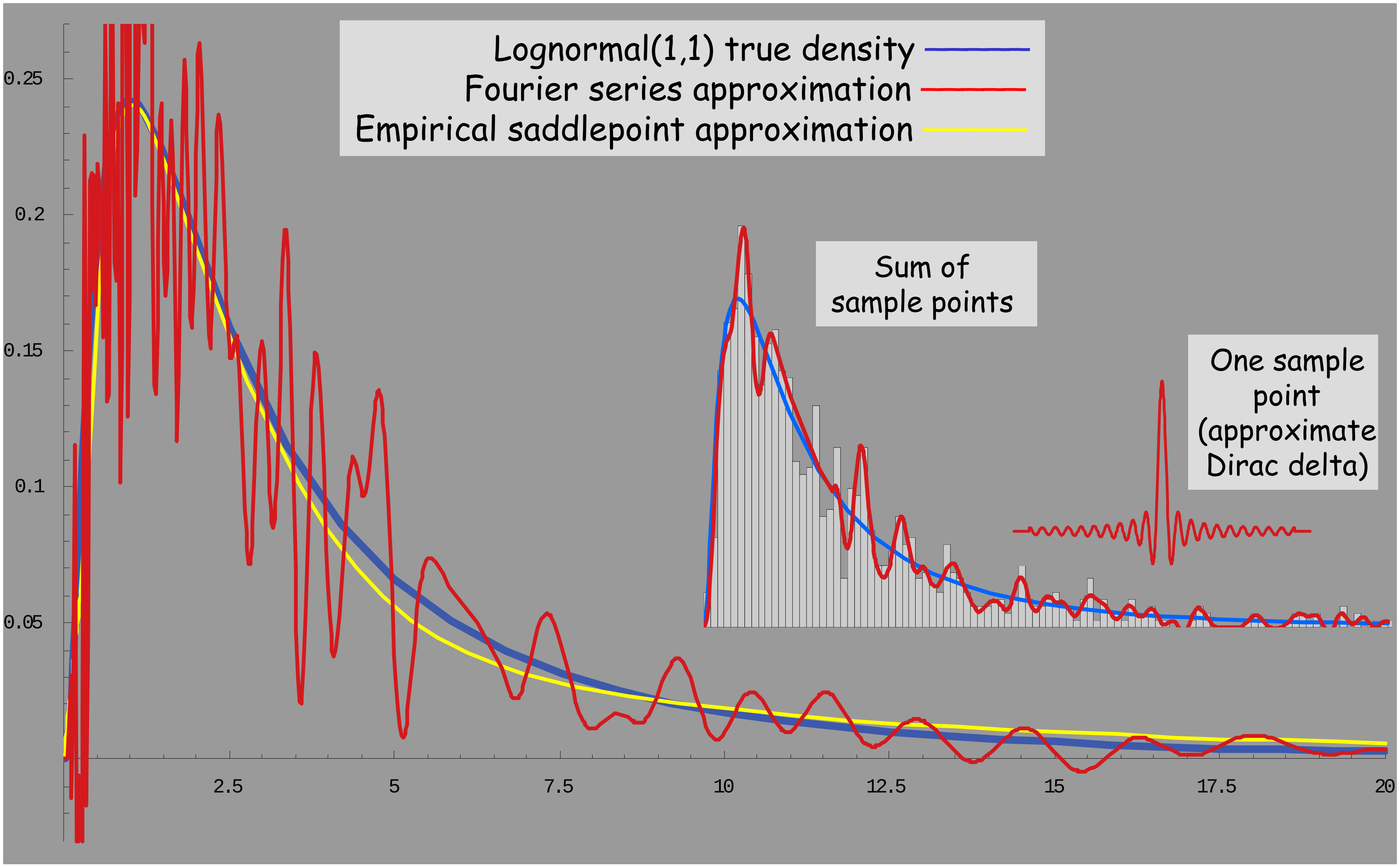
Use Euler summation to accelerate convergence:

$$f(t) \approx \sum_{m=0}^{M}\binom{M}{m}2^{-m}s_{N+m}(t).$$

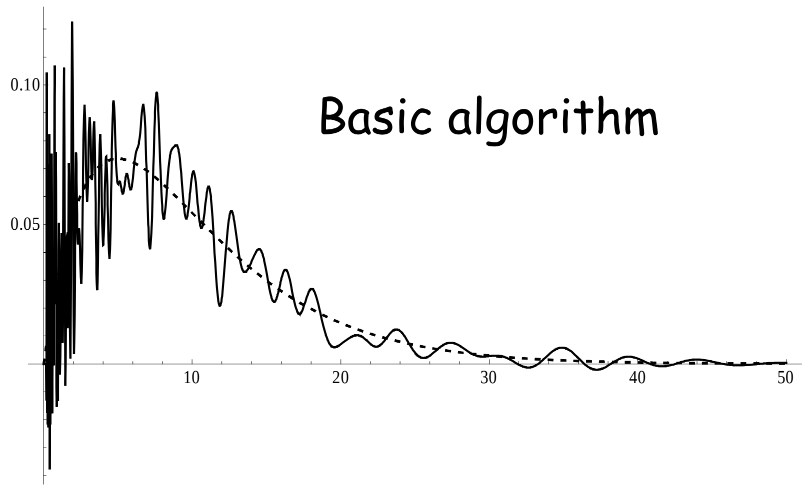
- Very accurate for exact transforms of continuous densities
- Less so for empirical transforms, improves with complexity of flowgraph

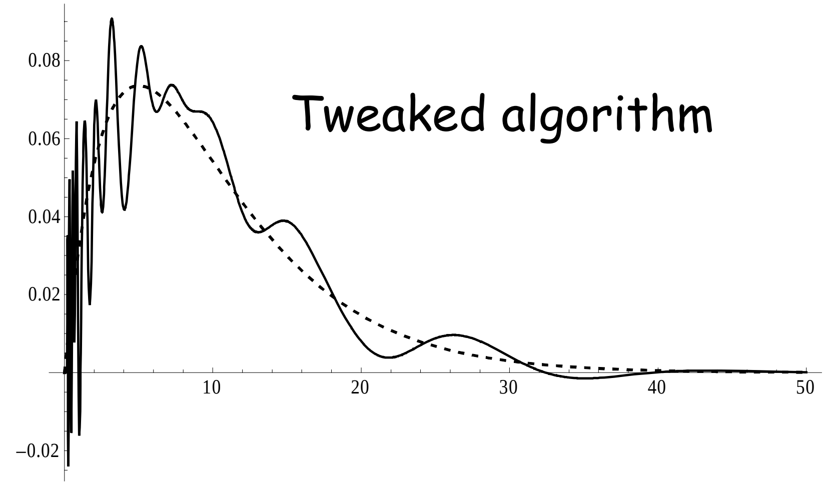
J. Abate and W. Whitt, W. (1992), "The Fourier-series method for inverting transforms of probability distributions," *Queueing Systems* 10, 5-88.

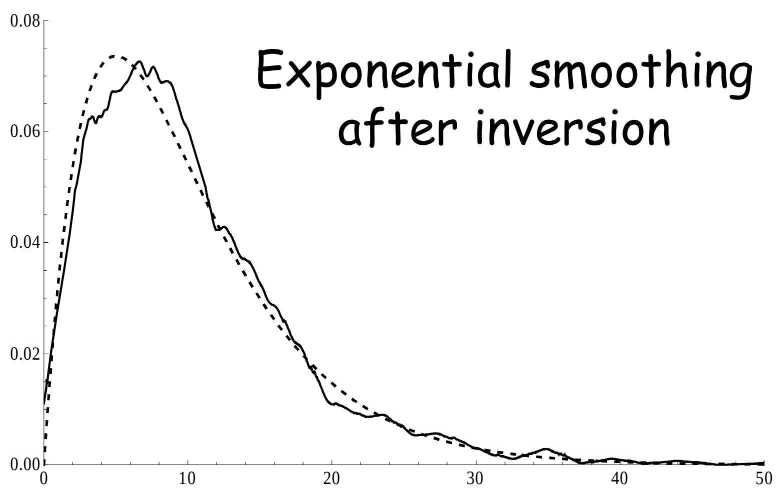
Inversion of an empirical transform

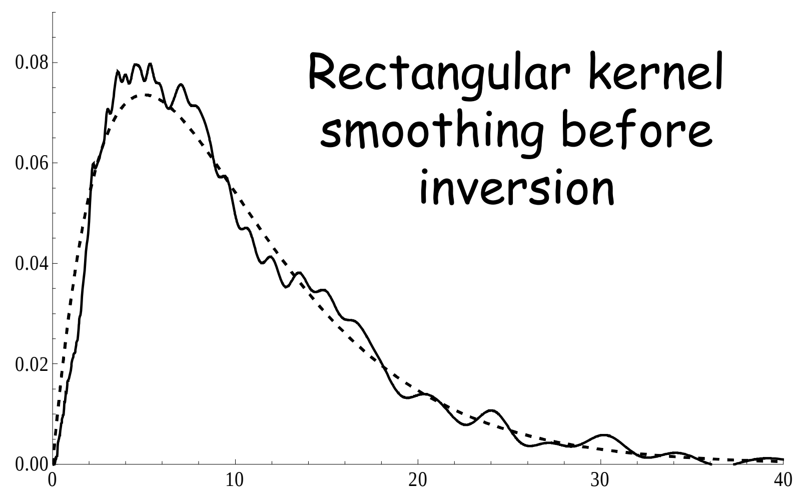


Smoothing the Fourier series approximation







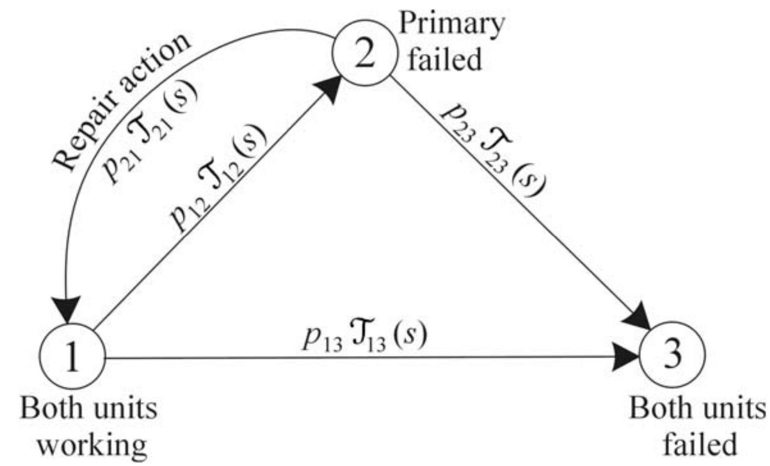


Example – Three-state repairable redundant system

- Each unit has independent failure distribution $\exp(\lambda_1)$ so $1 \to 2$ transition ~ $\exp(2\lambda_1)$ (assume P[simultaneous failure] = 0)

- $1 \to 3$ transition ~ $\exp(\lambda_2)$ (common cause failure, e.g., power outage)

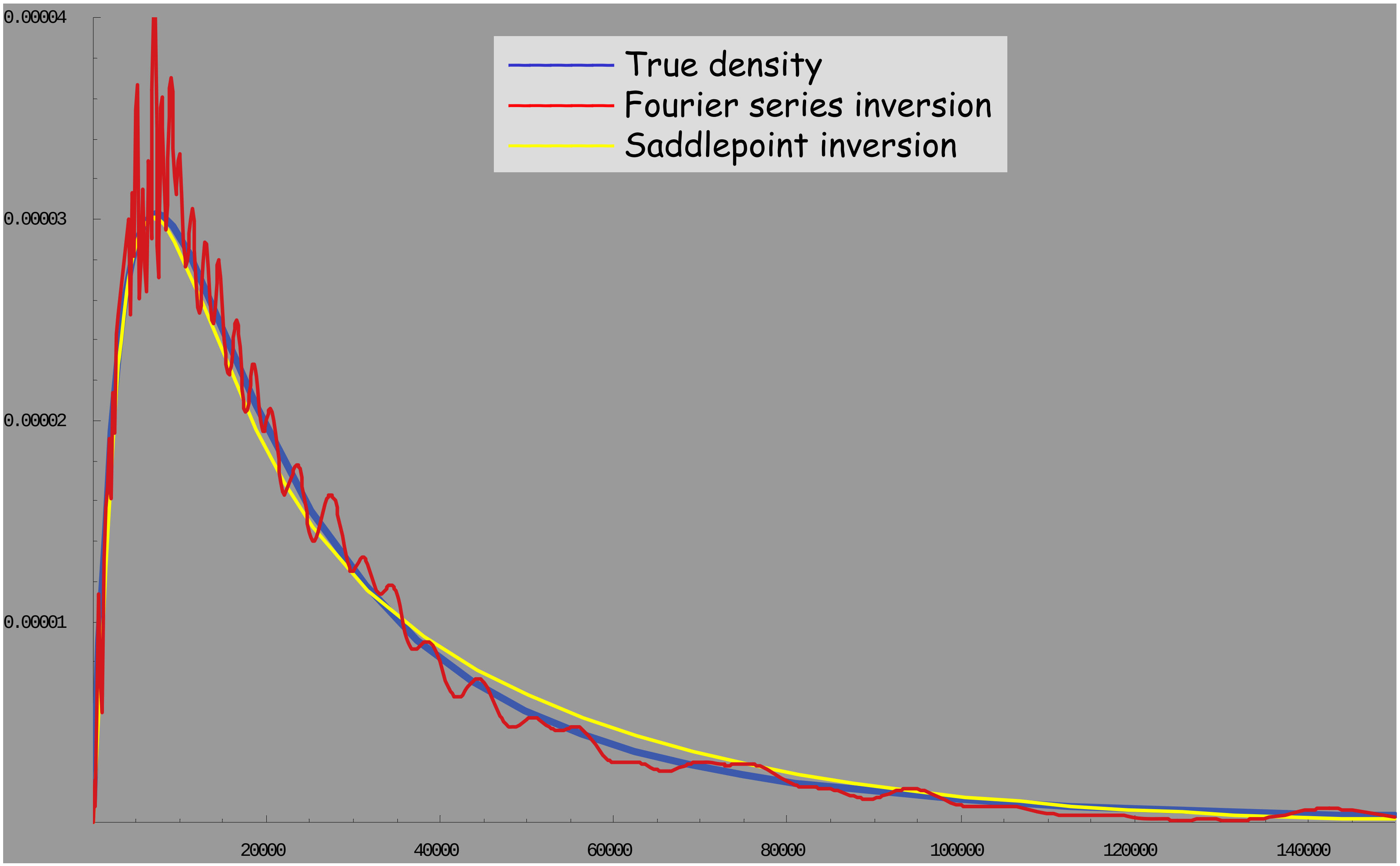
- $2 \to 1$ transition (failed unit repaired) ~ gamma(α, β)



- Sample sizes 270, 25, 247, 23 for $1 \to 2$, $1 \to 3$, $2 \to 1$, $2 \to 3$ (realistic for well-controlled field operation of one system over 1-2 years, or data from multiple systems)

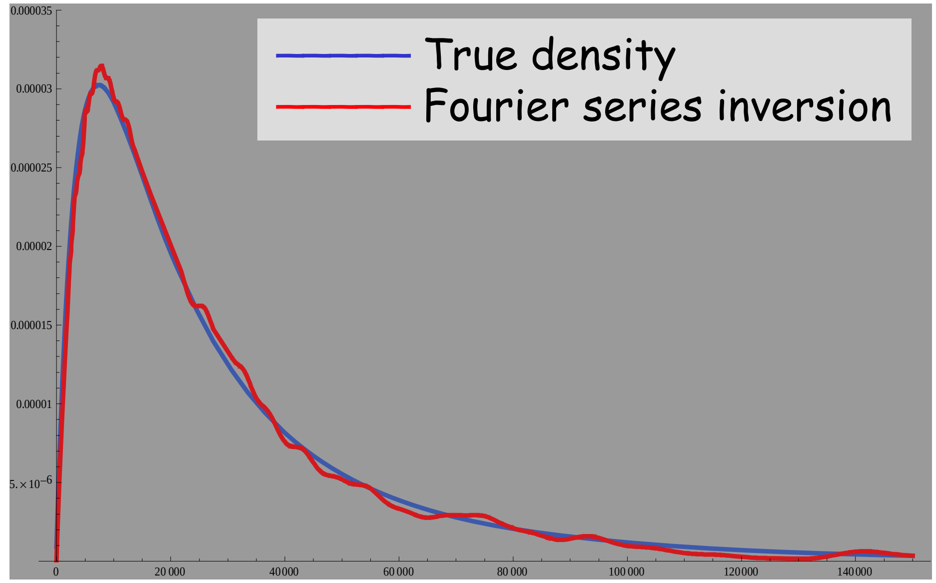
- Interested in $1 \to 3$ first passage $\equiv$ time to total system failure

Three-state repairable redundant system

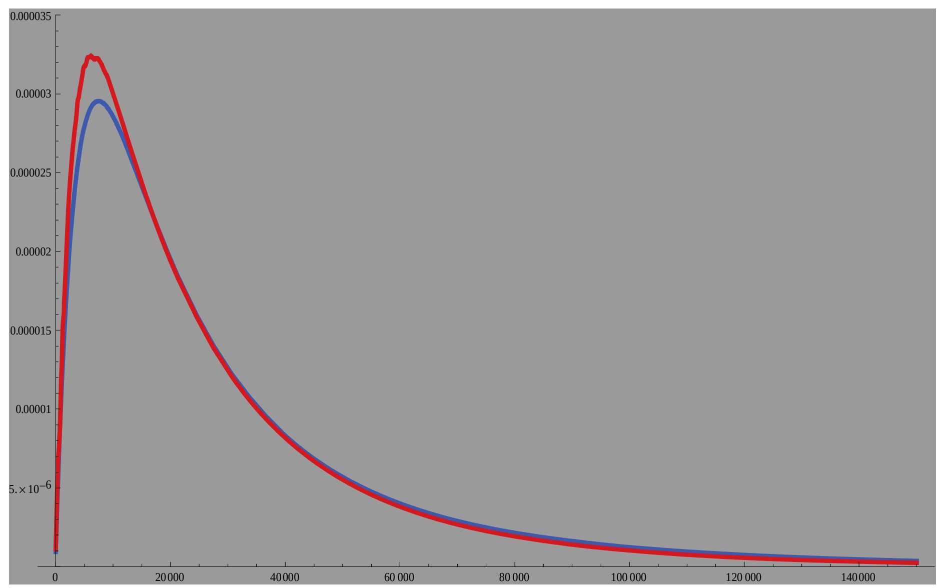


Repairable redundant system - alternatives

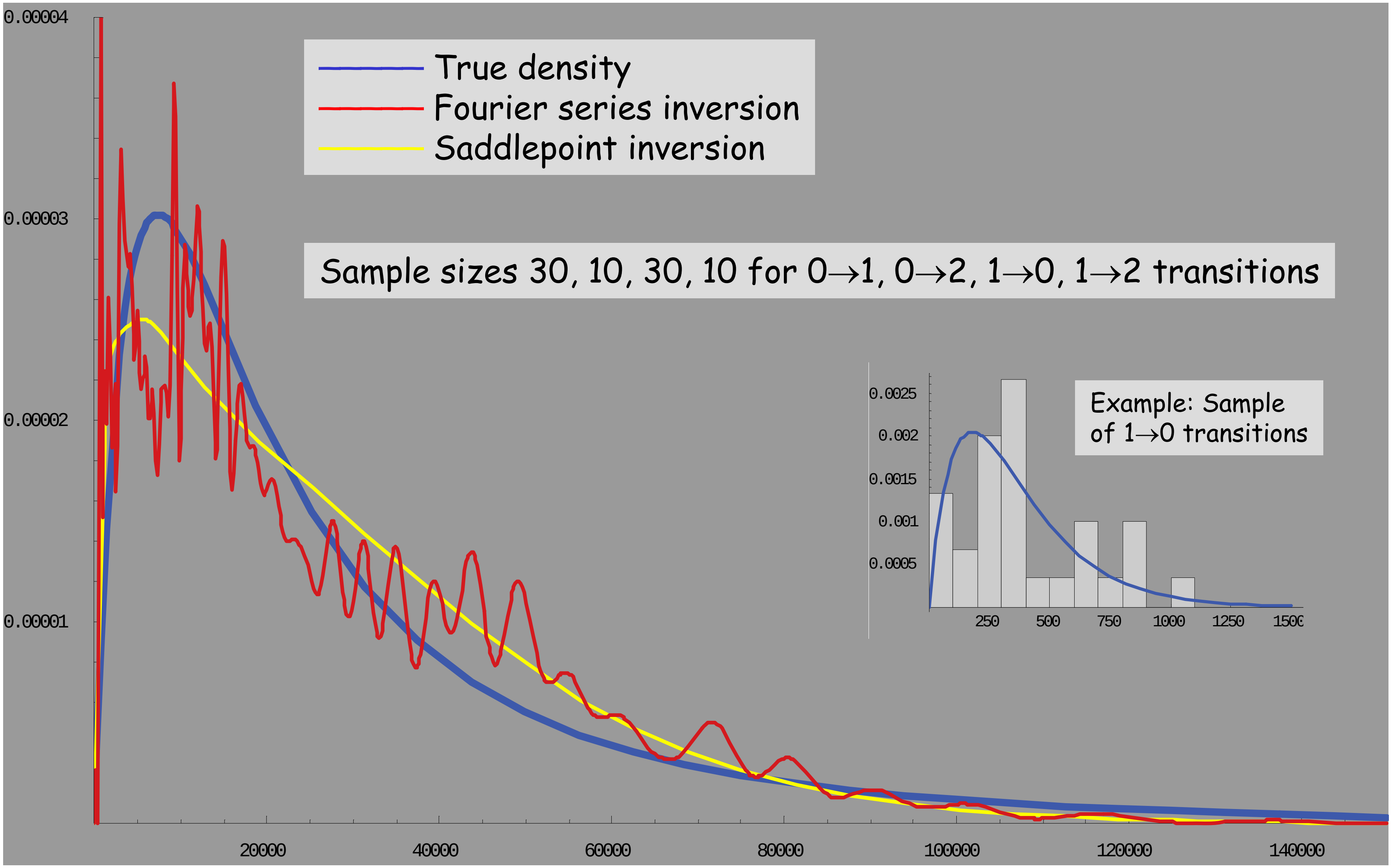
Exponential smoothing of Fourier series inversion points



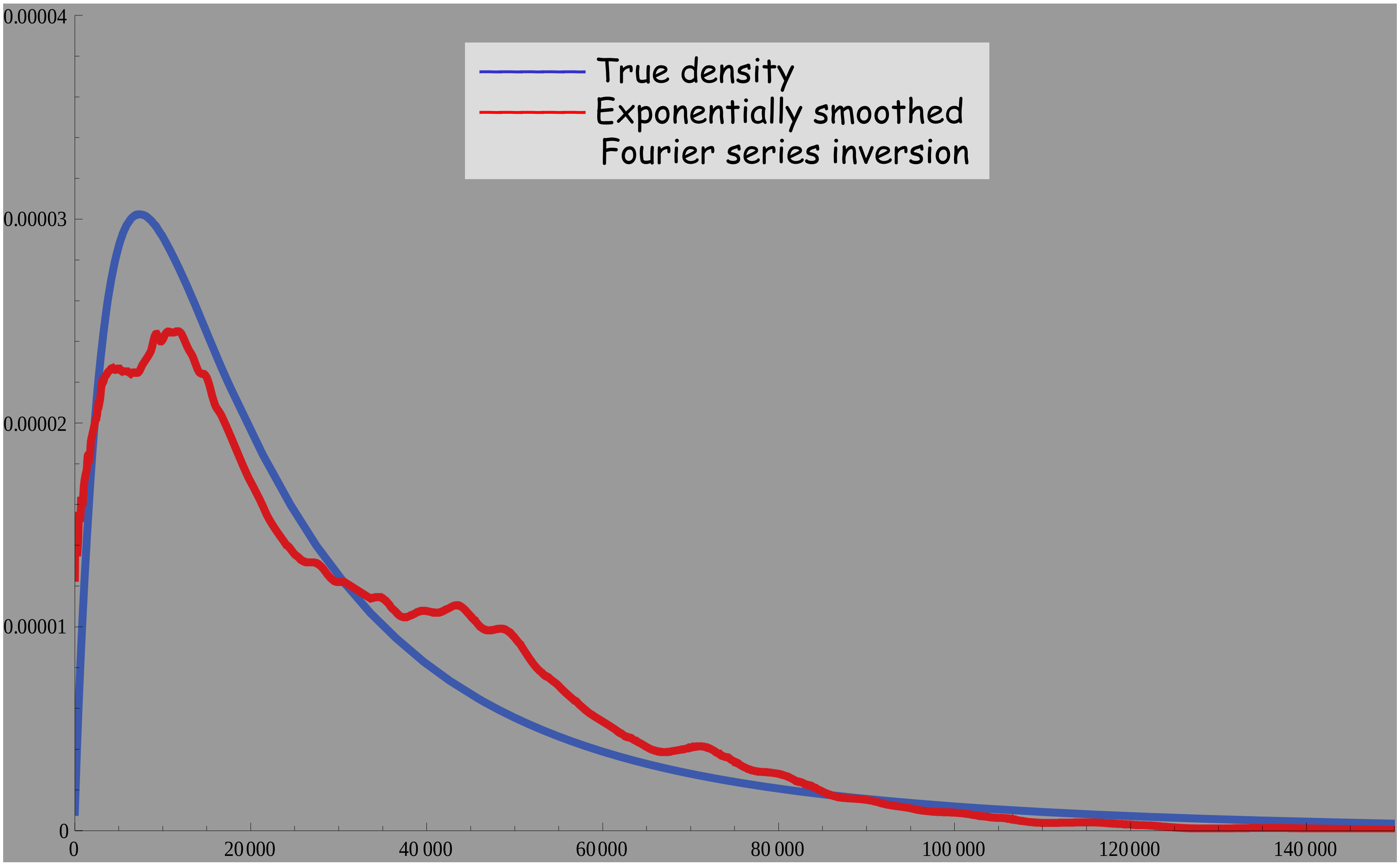
Semiparametric model:
- Empirical transforms for 0→1 and 1→0 transitions
- 0→2 and 1→2 transitions parametric exp(λ), λ estimated by maximum likelihood



Repairable redundant system – small sample

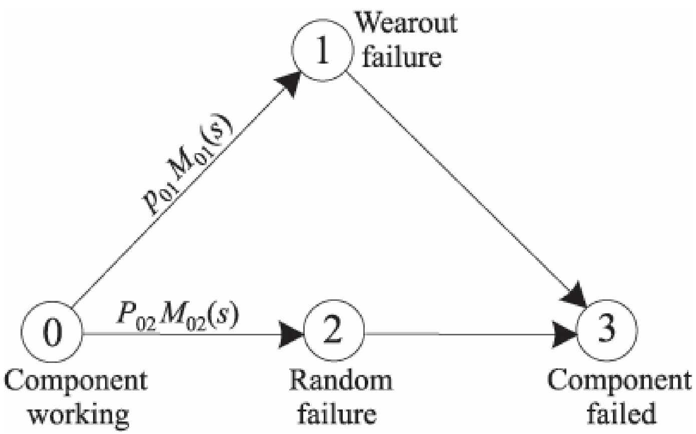


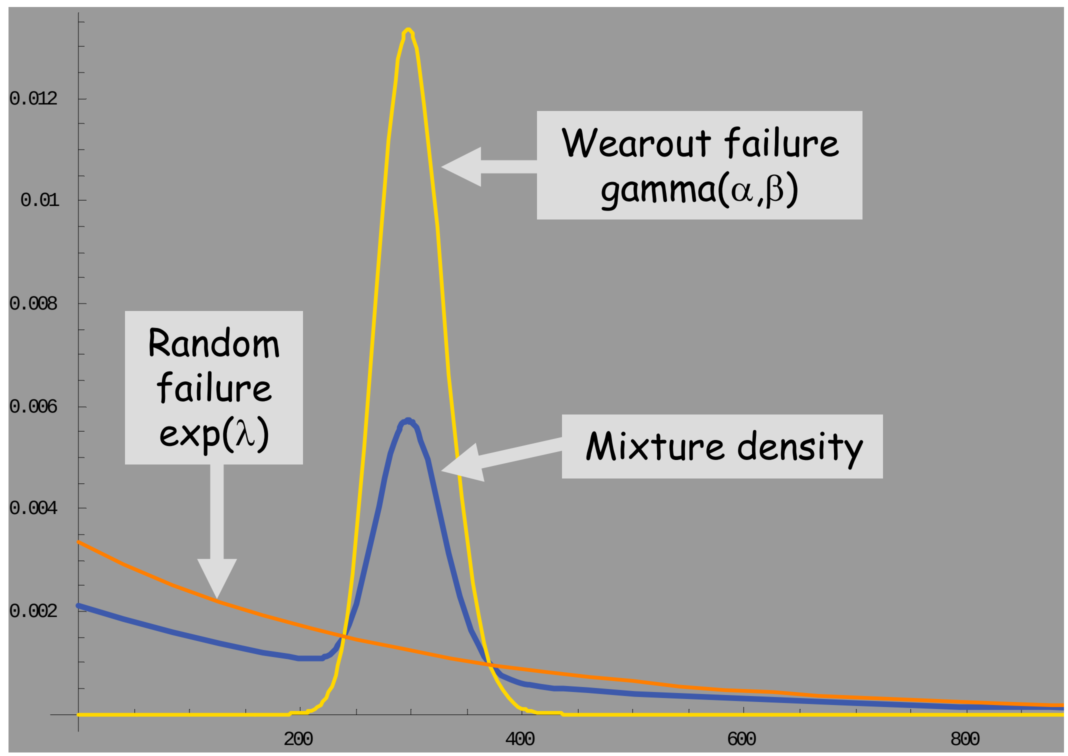
Small sample with exponential smoothing



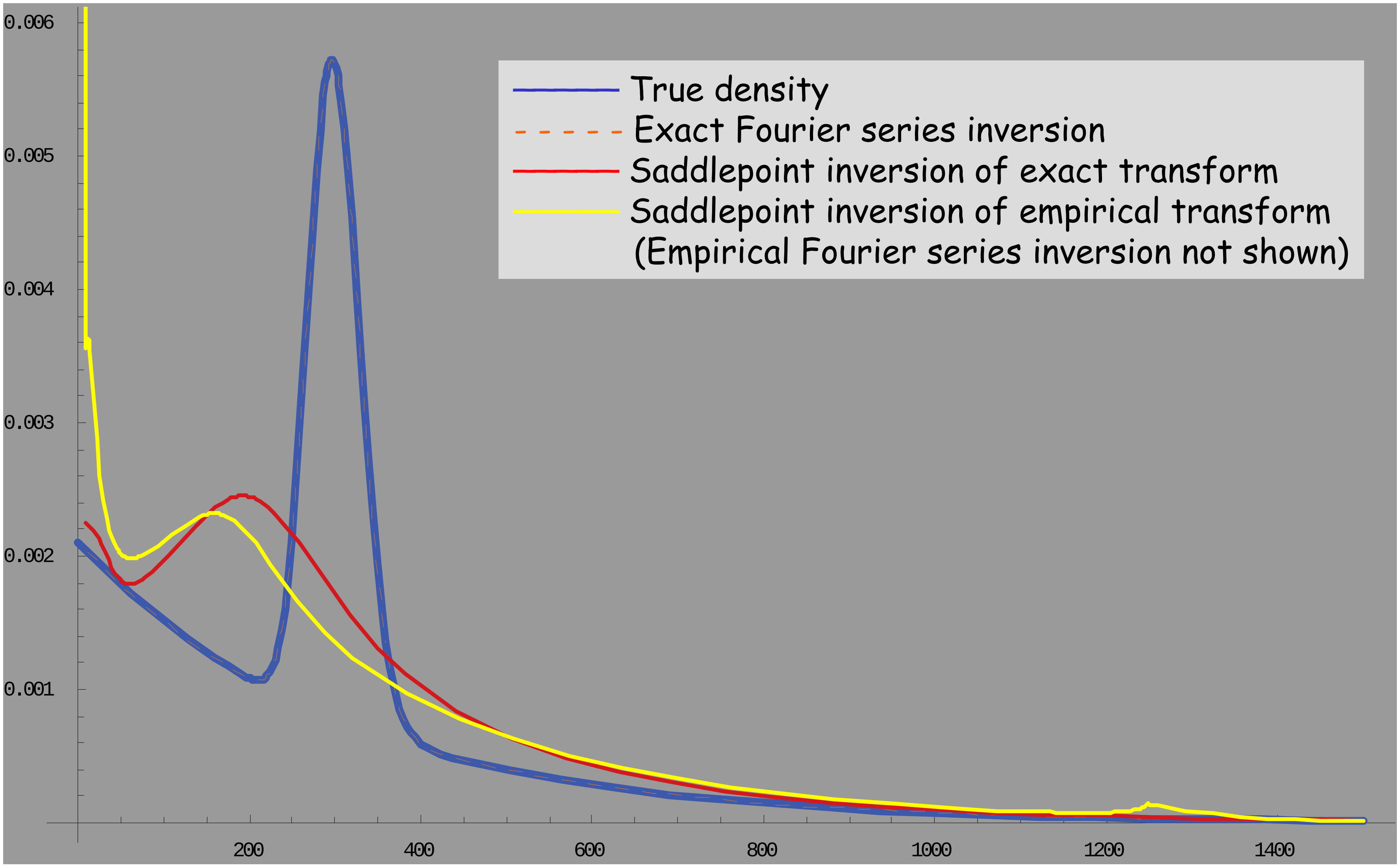
Example – Wearout and random failure modes

- Complex component with two failure modes
- Challenging case for saddlepoint method
- Flowgraph and densities:



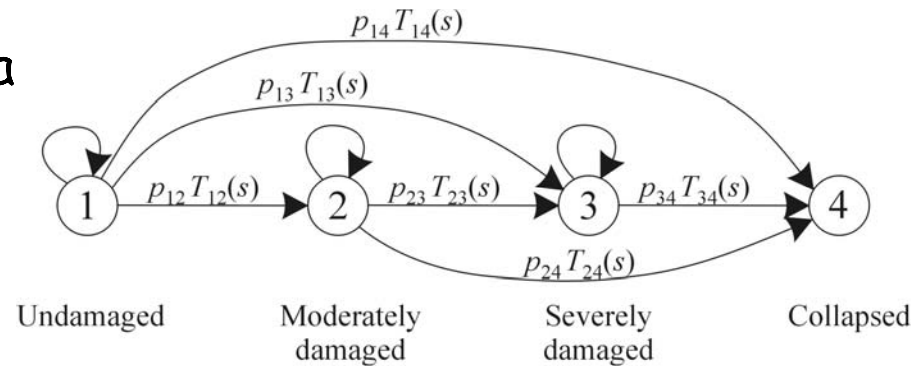


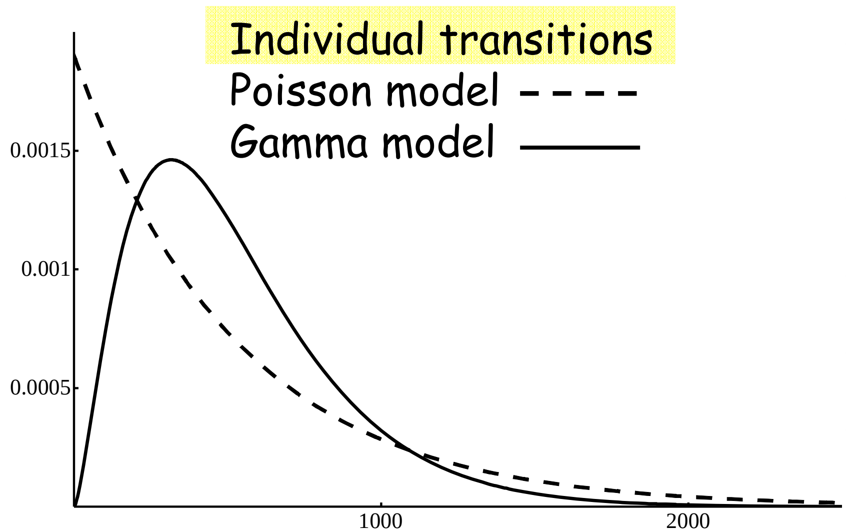
Example – Wearout and random failure modes

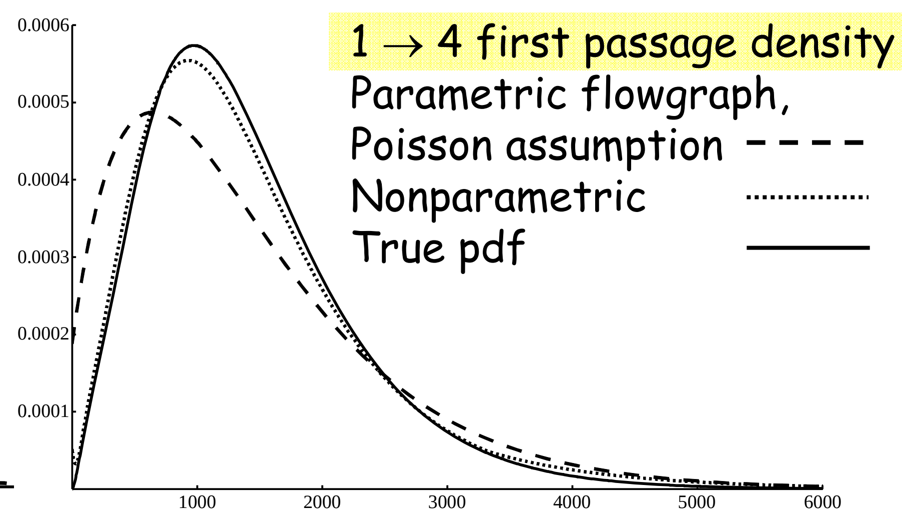


Detecting a model error

Model for cumulative damage to a structure from repeated seismic shocks. Transitions modeled as a homogeneous Poisson process.







D. H. Collins & A. V. Huzurbazar, "System reliability and safety assessment using nonparametric flowgraph models," *Journal of Risk and Reliability*, to appear.

Pros & cons of nonparametric method

- How much confidence do we have in a particular parametric model?
  - Non-parametric result can be worse, or better
  - Small-sample results may not be very useful
- Computational complexity is much greater with non-parametric method (probably improvable)
- Empirical transform always exists (e.g., for lognormal)
- Non-parametric method is a valuable adjunct even if a parametric model is available
  - Reinforcement if confirmatory
  - If not, indicates need for further analysis

Summary and conclusions

- Illustrated nonparametric method for statistical flowgraphs
  - Realistic examples for reliability scenarios
  - Useful in itself, and for validation of parametric results
- Further work:
  - More efficient algorithms (e.g., for numerical integration of empirical saddlepoint)
  - Adaptive smoothing for Abate & Whitt inversion
  - Handling of left/interval censored/truncated data
  - Large-sample properties of nonparametric approximations

Thanks for your attention . . .
Questions?